Cutting Multiparticle Correlators Down to Size

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with Eric Metodiev and Jesse Thaler, to appear soon

BOOST 2019 – MIT
July 24, 2019

Multiparticle Correlators

Sums of products of energies (transverse momenta) and angles

Definition of energy factor and pairwise angular distance

$$z_i = \frac{p_{Ti}}{\sum_j p_{Tj}} \qquad \theta_{ij}^2 = 2 n_i^\mu n_{j\mu} = 2 \frac{p_i^\mu}{p_{Ti}} \frac{p_{\mu j}}{p_{Tj}} \simeq (y_i - y_j)^2 + (\phi_i - \phi_j)^2$$

↑ central, narrow jet approximation

Graphs represent correlators

vertex ↔ energy factor

$$\longleftrightarrow \sum_{i_j=1}^{M} z_{i_j}$$

edge ↔ pairwise angle

$$\longleftrightarrow \theta_{i_k i_l}$$

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Ubiquitous observables at the LHC

Mass

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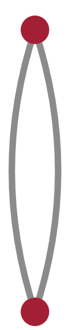
$\frac{1}{2}$ 

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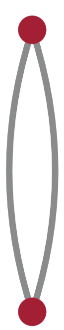
[Larkoski, Salam, Thaler, 1305.0007; Larkoski, Moult, Neill, 1409.6298]

$\frac{1}{2}$ 

Used for multi-prong tagging, typically in ratios , D_2, C_2, C_3, etc.

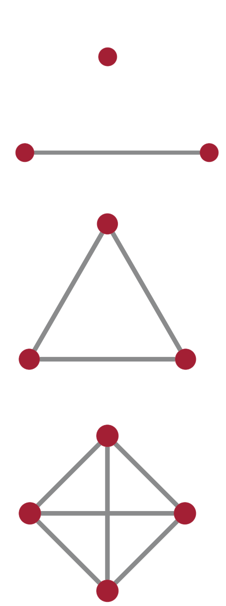
Generalized ECFs also useful (angular part not monomial)

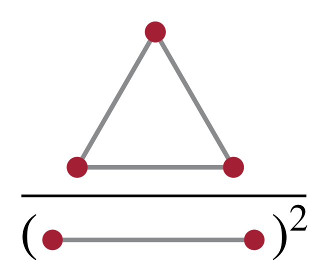
$N = 1$:

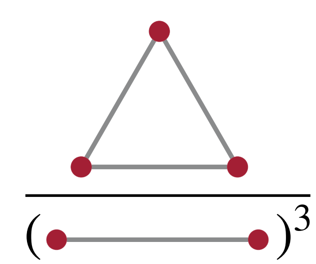
$N = 2$:

$N = 3$:

$N = 4$:



C_2 : 

D_2 : 

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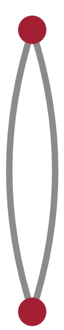
[PTK, Metodiev, Thaler, 1712.07124]

$\frac{1}{2}$ 

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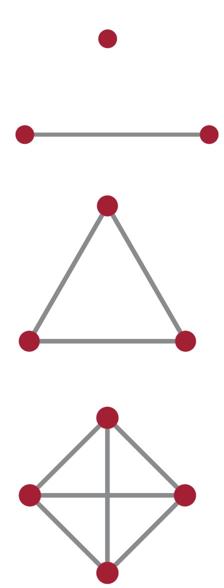
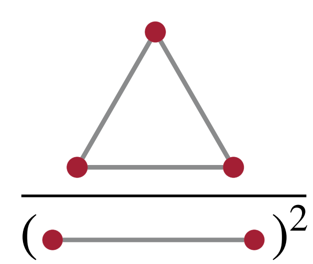
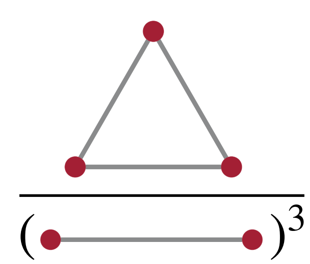
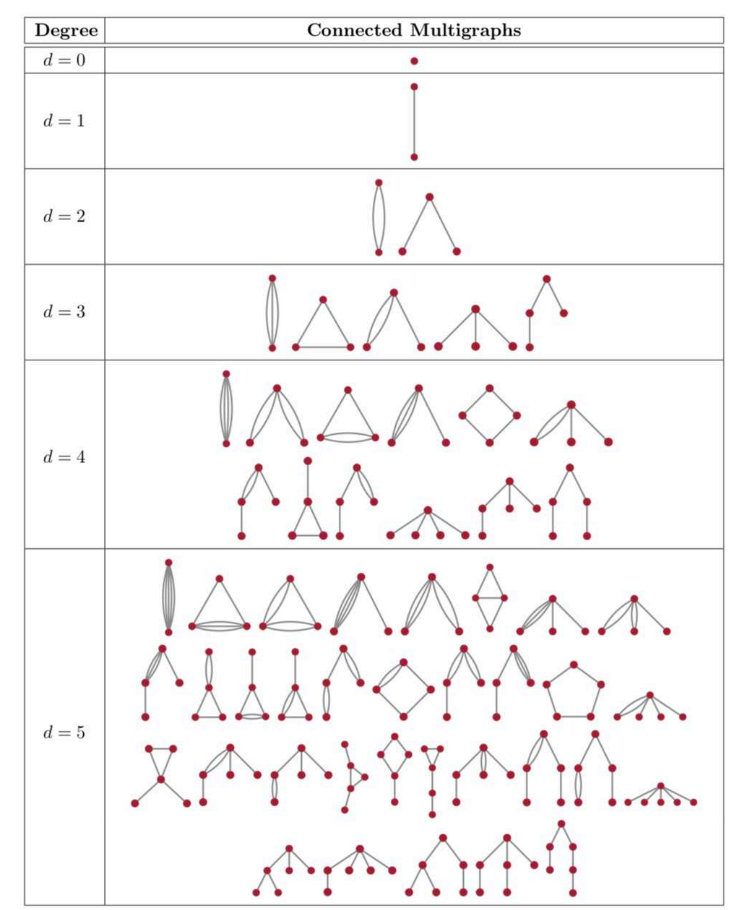


C_2 : 

D_2 : 

Linear basis of all IRC-safe observables

$$\mathcal{O} = \sum_G s_G \mathrm{EFP}_G$$



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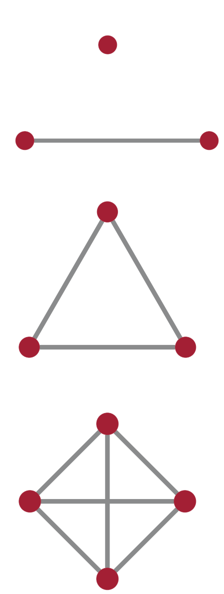

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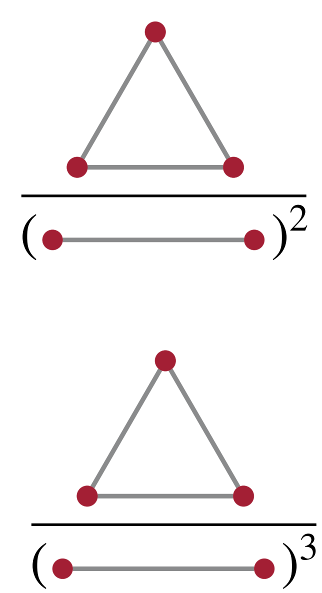

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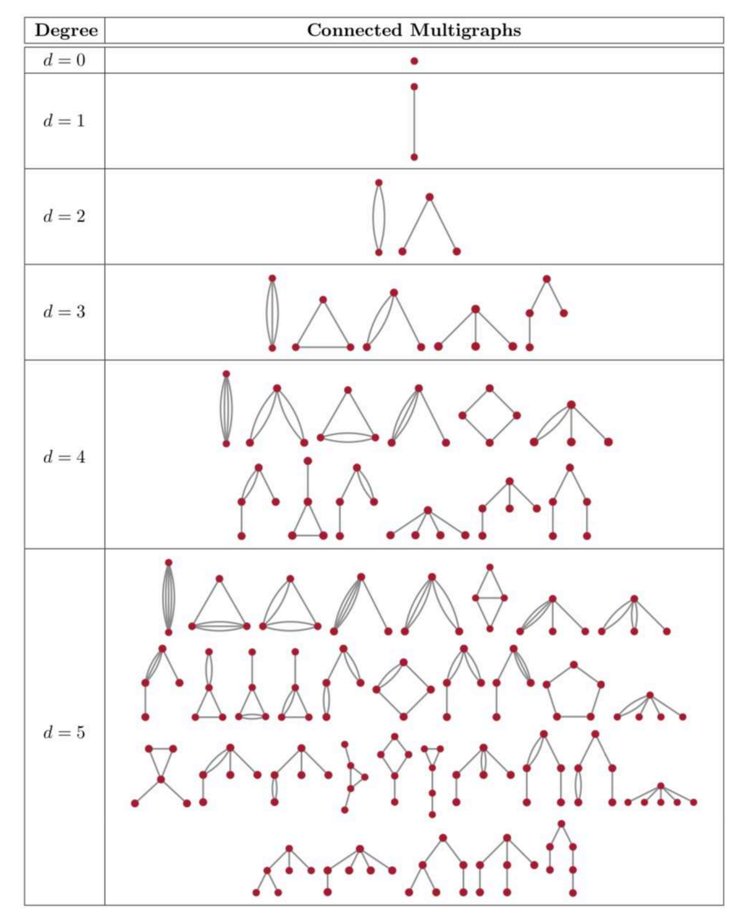


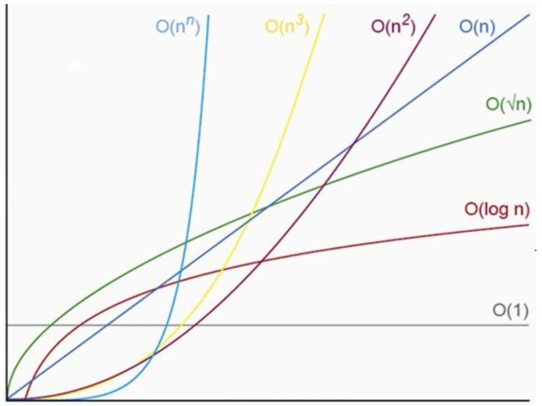
Mass also calculated as

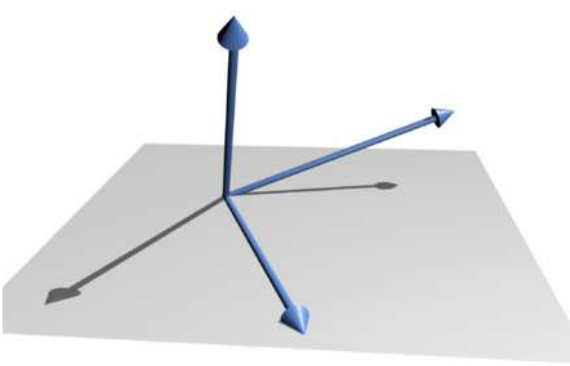
$$\left(\sum_{i=1}^{M} p_i^\mu\right)^2$$

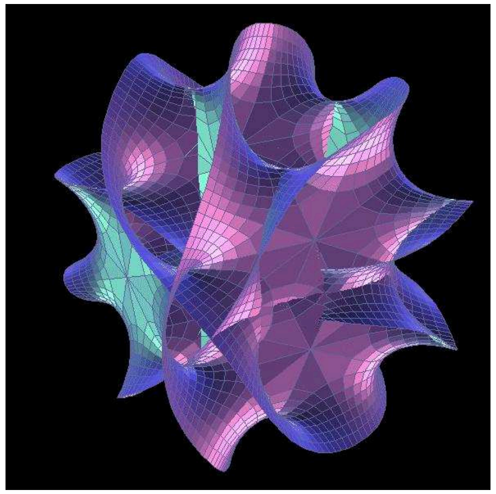
which is $O(M)$ to compute

What else is $O(M)$?

Outline







Experiment

Computational Complexity

Multiparticle correlators are $\mathcal{O}(M^N)$ to compute in general

Many can actually be computed in $\mathcal{O}(M)$

Linear Tensor Identities

Multiparticle correlators exhibit mysterious linear redundancies

All redundancies understood via cutting graphs

Counting Superstring Amplitudes

Counting independent kinematic polynomials difficult

Immediate enumeration through multigraphs

Theory

Computational Complexity – BOOST 2018

Naive computation complexity of an energy correlator is $\mathcal{O}(M^N)$

EnergyCorrelator `fjcontrib` solution:

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```
// if N > 5, then throw error
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    throw Error("EnergyCorrelator is only hard coded for N = 0,1,2,3,4,5");
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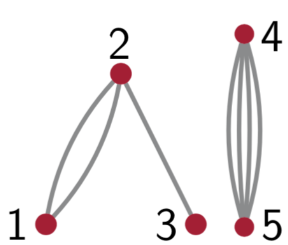
Variable elimination (VE) algorithm: $\mathcal{O}(M^\chi)$, $\chi \lesssim N$

Disconnected is product of connected



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VE find clever parentheses placement to minimize computation

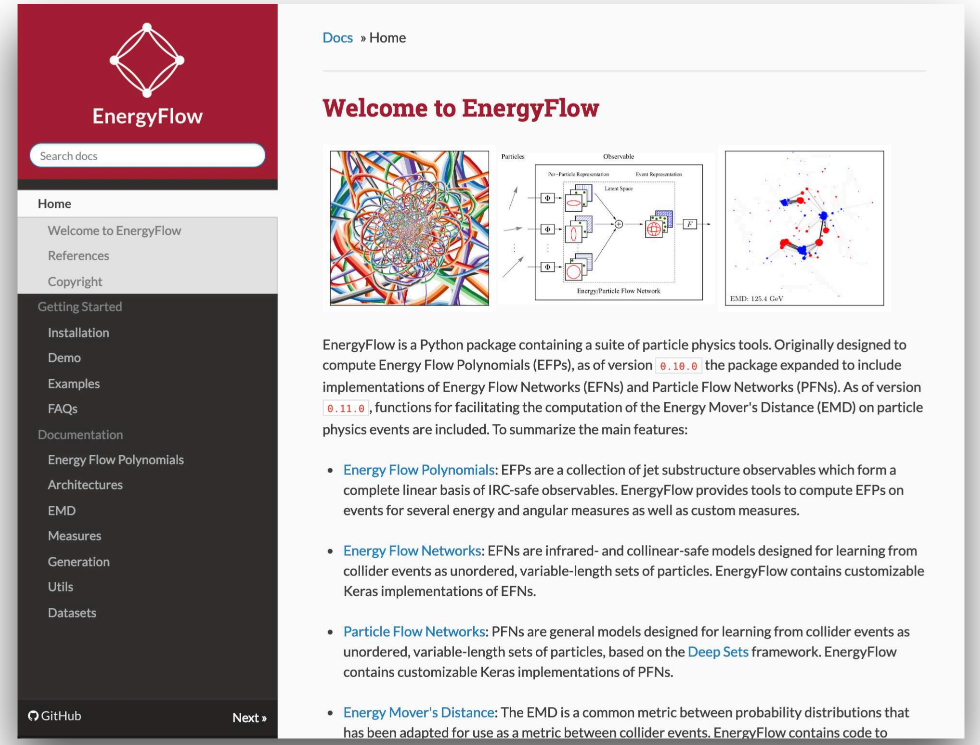
$$= \underbrace{\sum_{i_1=1}^{M}\sum_{i_2=1}^{M}\sum_{i_3=1}^{M}\sum_{i_4=1}^{M}\sum_{i_5=1}^{M}\sum_{i_6=1}^{M}\sum_{i_7=1}^{M}\sum_{i_8=1}^{M} z_{i_1} z_{i_2} z_{i_3} z_{i_4} z_{i_5} z_{i_6} z_{i_7} z_{i_8} \prod_{j=2}^{7} \theta_{i_1 i_j}}_{\mathcal{O}(M^8)}$$

$$= \underbrace{\sum_{i_1=1}^{M} z_{i_1} \left(\sum_{i_2=1}^{M} z_{i_2} \theta_{i_1 i_2}\right)^7}_{\mathcal{O}(M^2)}$$

All tree graphs become $\mathcal{O}(M^2)$

$\chi = N$ iff G is complete graph, ECFs still slow

[PTK, Metodiev, Thaler, 1712.07124]



https://energyflow.network

Computational Complexity – BOOST 2018

BOOST 2019

Can we do better – perhaps $\mathcal{O}(M)$ as for mass?

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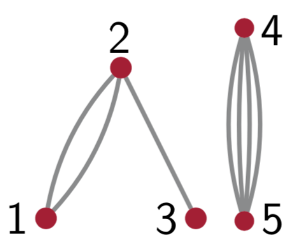

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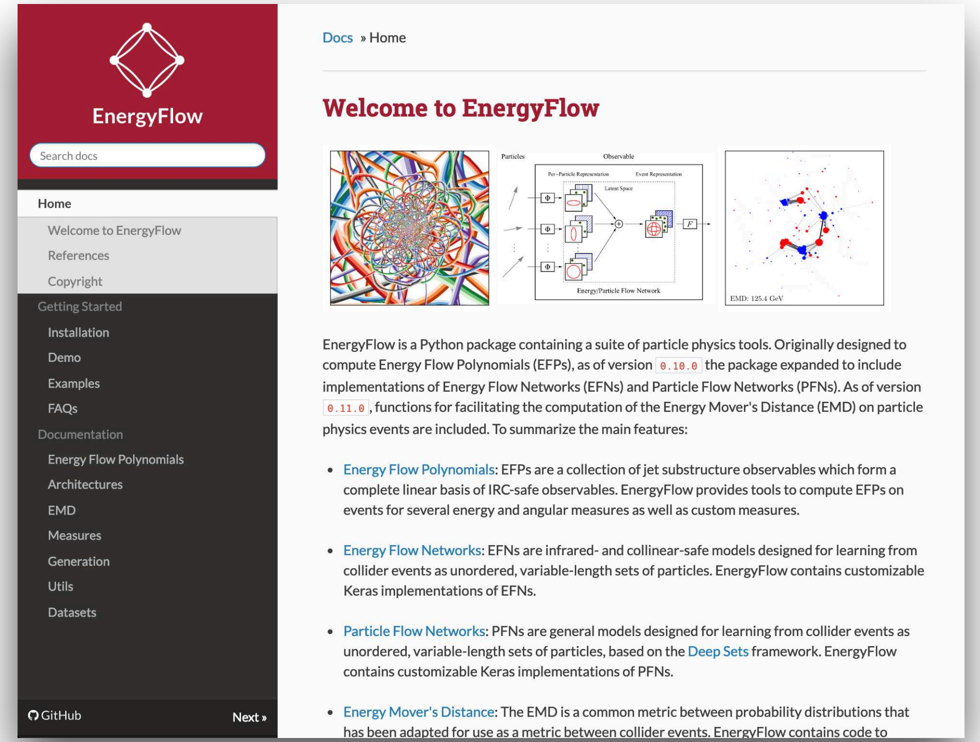

https://energyflow.network

Energy Flow Moments (EFMs)

[PTK, Metodiev, Thaler, *to appear soon*]

$\theta_{ij} = \sqrt{2 n_i^\mu n_{j\mu}}$ $\beta = 2$ removes square root

Factors of n_i^μ can be organized in optimal way

EFM$_v$ is a totally symmetric little group tensor

$$\mathcal{I}^{\mu_1 \cdots \mu_v} = \sum_{i=1}^{M} z_i n_i^{\mu_1} \cdots n_i^{\mu_v}$$

v	0	1	2	3	4	5	6
$n_{\text{components}}^{(d=4)}$	1	4	10	20	35	56	84

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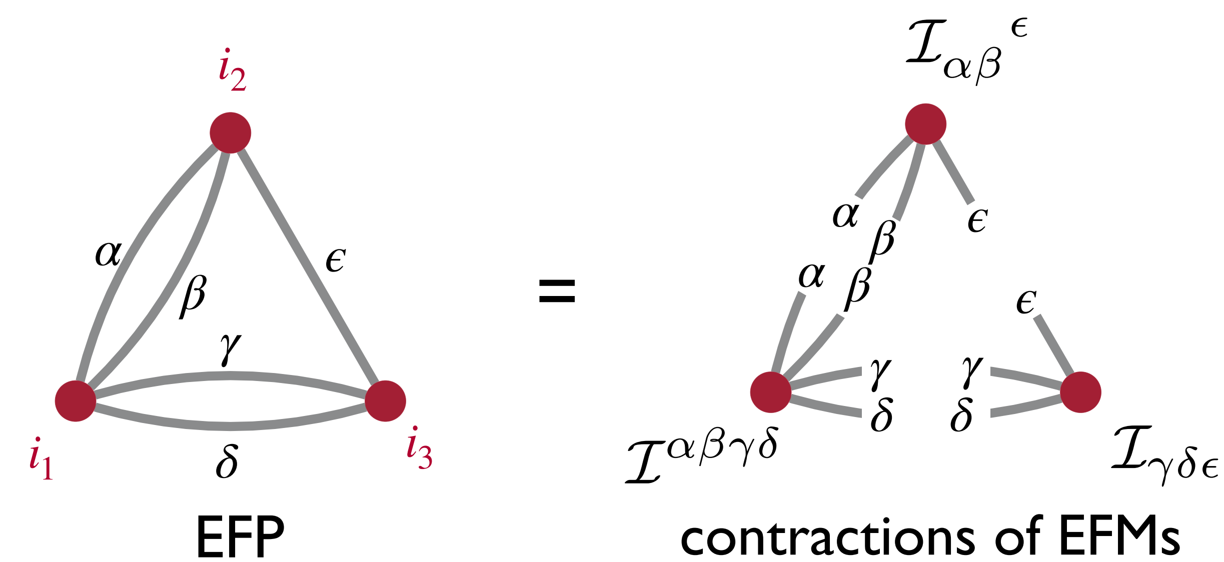
EFMs result from cutting edges of EFP graph



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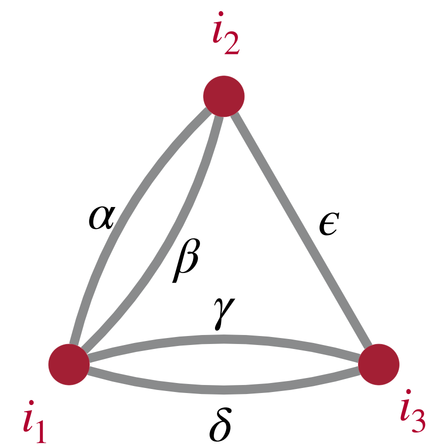
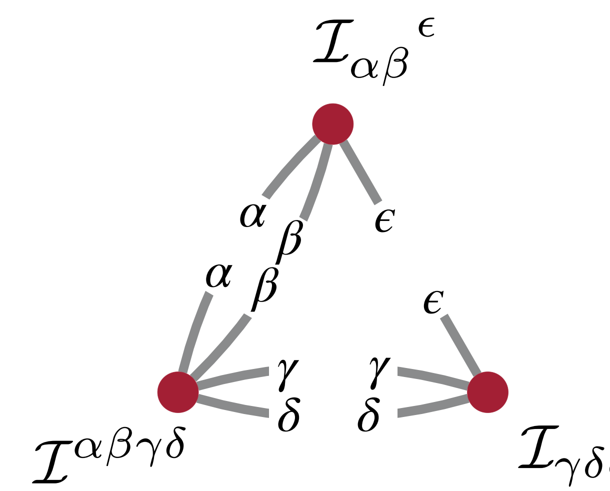
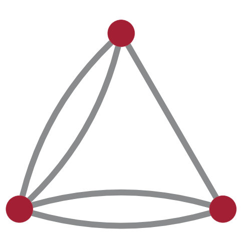


EFP

=



contractions of EFMs



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$$= 2^5 \underbrace{\left(\sum_{i_1=1}^{M} z_{i_1} n_{i_1}^\alpha n_{i_1}^\beta n_{i_1}^\gamma n_{i_1}^\delta \right)}_{\mathcal{I}^{\alpha\beta\gamma\delta}} \underbrace{\left(\sum_{i_2=1}^{M} z_{i_2} n_{i_2 \alpha} n_{i_2 \beta} n_{i_2}^\epsilon \right)}_{\mathcal{I}_{\alpha\beta}{}^{\epsilon}} \underbrace{\left(\sum_{i_3=1}^{M} z_{i_3} n_{i_3 \gamma} n_{i_3 \delta} n_{i_3 \epsilon} \right)}_{\mathcal{I}_{\gamma\delta\epsilon}}$$

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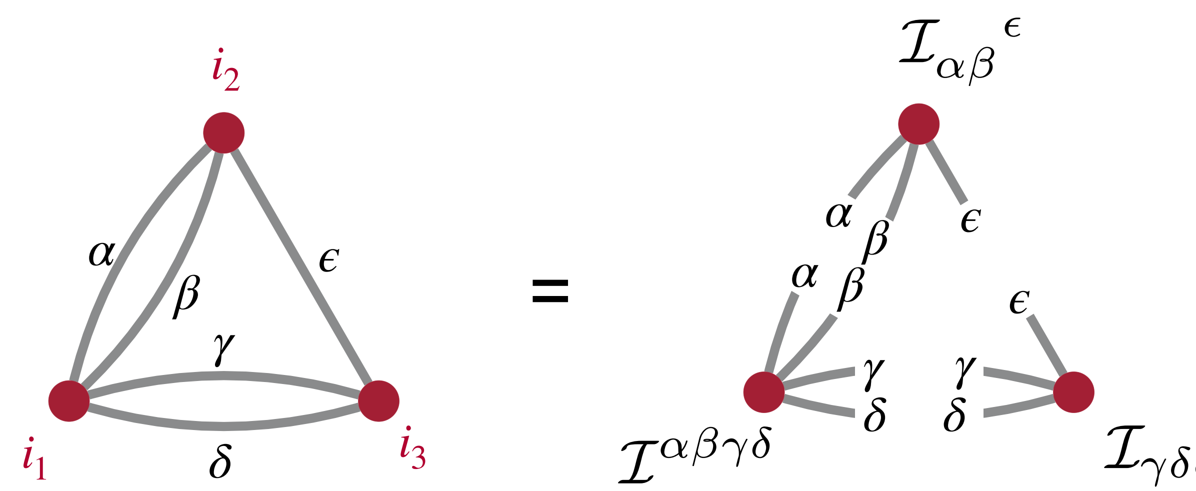
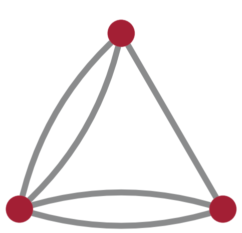


EFP contractions of EFMs

All $\beta = 2$ EFPs are $\mathcal{O}(M)$

ECF$_N^{(\beta=2)}$ are all $\mathcal{O}(M)$

$D_2^{(\beta=2)}$, $C_2^{(\beta=2)}$ are $\mathcal{O}(M)$



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See detailed derivation in backup

Linear Tensor Identities

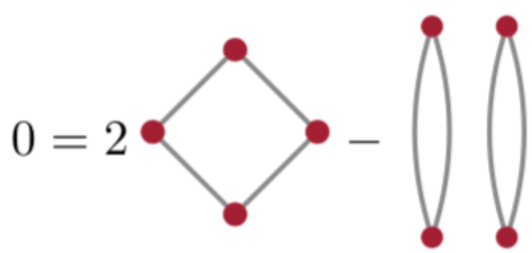
Linear redundancies among EFPs are troublesome

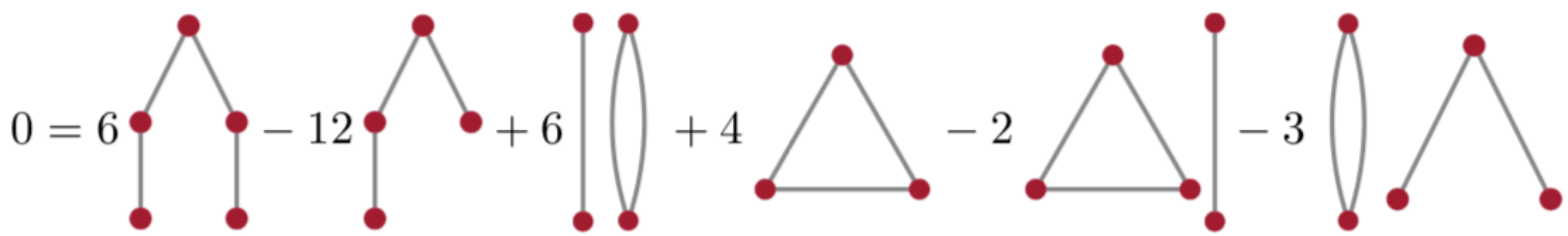
Studying coefficients of linear fit difficult

$$\mathcal{O} = \sum_G s_G \mathrm{EFP}_G$$

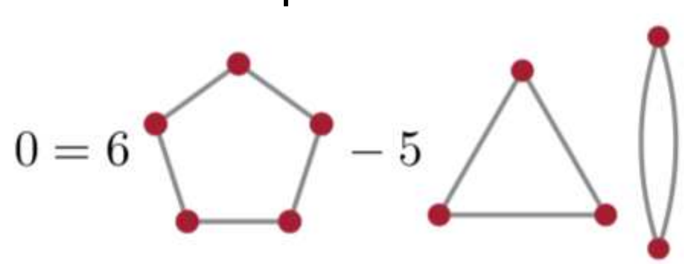
Examples of redundancies

in 3 or fewer spacetime dimensions





in 4 or fewer spacetime dimensions



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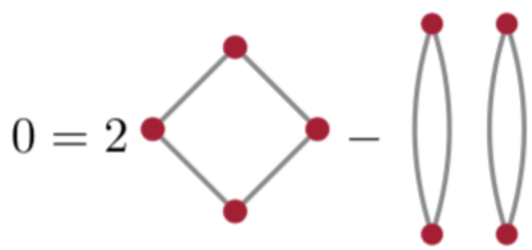

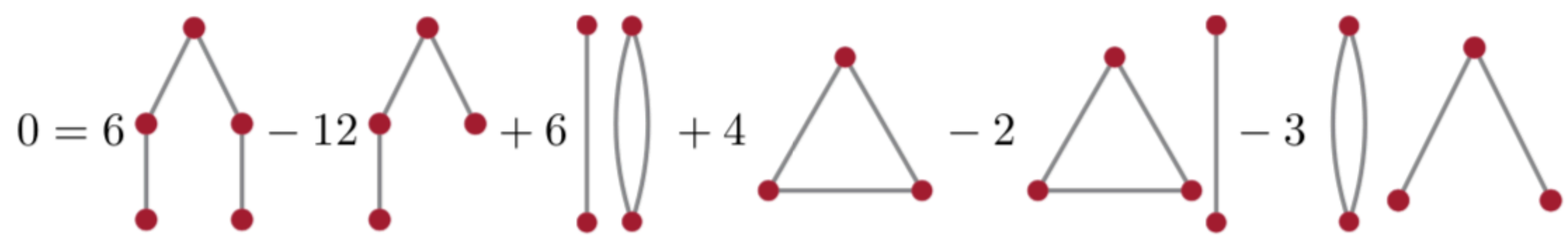

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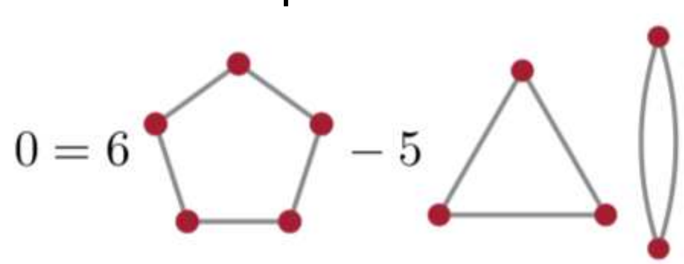

Tensor Identity Recipe

Consider tensor over *n* dimensional vector space

Antisymmetrize *m* > *n* indices

Result is zero because any assignment of *n* possible values to *m* slots has a repetition

$$T^{a_1 \cdots a_k}_{b_1 \cdots b_\ell [c_1 \cdots c_m]} = 0$$

Bonus: all tensor identities up to ones governed by existing symmetries take above form

[Sneddon, Journal of Mathematical Physics]

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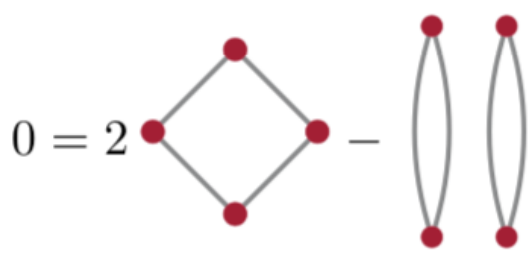
Studying coefficients of linear fit difficult

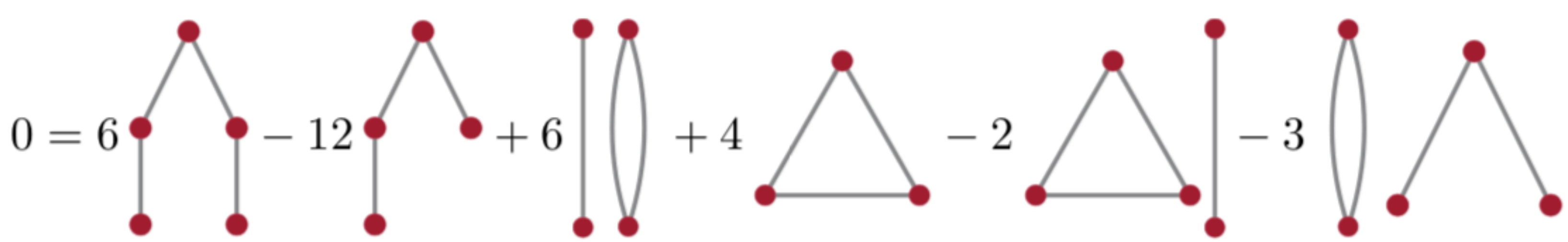
$$\mathcal{O} = \sum_G s_G \mathrm{EFP}_G$$

Examples of redundancies

in 3 or fewer spacetime dimensions



$$\iff \quad 0 = \mathcal{I}^{\beta}_{[\alpha}\mathcal{I}^{\gamma}_{\beta}\mathcal{I}^{\delta}_{\gamma}\mathcal{I}^{\alpha}_{\delta]}$$



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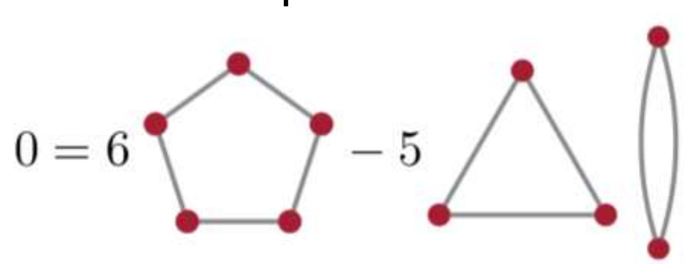
in 4 or fewer spacetime dimensions



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Tensor Identity Recipe

Consider tensor over *n* dimensional vector space

Antisymmetrize *m* > *n* indices

Result is zero because any assignment of *n* possible values to *m* slots has a repetition

$$T^{a_1\cdots a_k}_{b_1\cdots b_\ell[c_1\cdots c_m]} = 0$$

Bonus: all tensor identities up to ones governed by existing symmetries take above form

[Sneddon, Journal of Mathematical Physics]

Linear Tensor Identities

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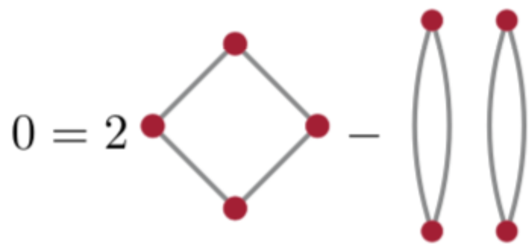
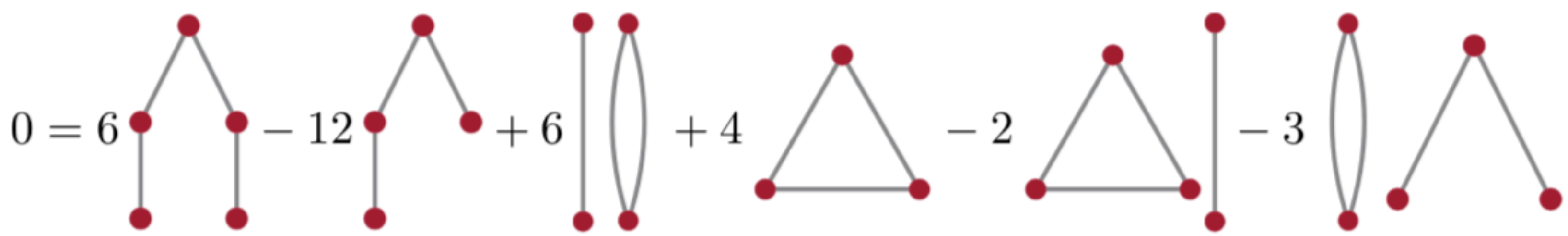


$\Longleftrightarrow$ $0 = \mathcal{I}^{\beta}_{[\alpha}\mathcal{I}^{\gamma}_{\beta}\mathcal{I}^{\delta}_{\gamma}\mathcal{I}^{\alpha}_{\delta]}$



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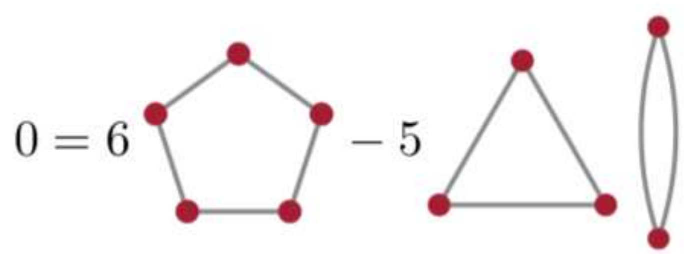

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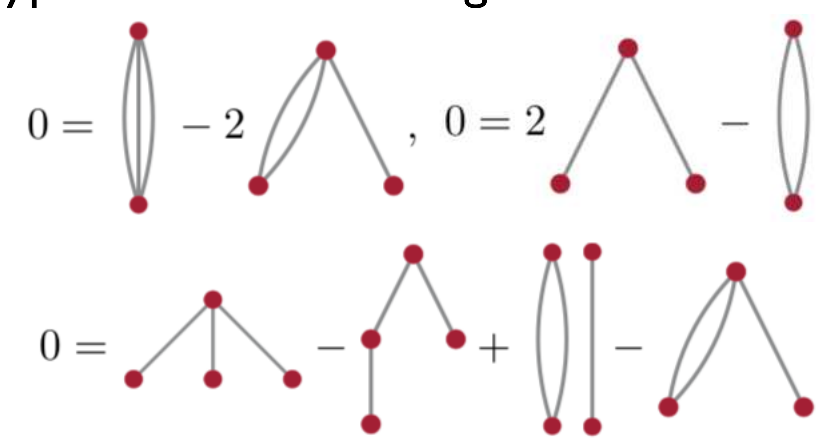
[Sneddon, Journal of Mathematical Physics]

Other types of identities – e.g. when M is small

$M \leq 2$



Could be useful in a partonic calculation, more in backup

Counting Superstring Amplitudes

Constructing a basis of amplitudes – how large is it?

[Boels, 1304.7918; OEIS A226919]

non-isomorphic multigraph

$\Updownarrow$

Q: What is the number of symmetric polynomials of degree d in kinematic variables $s_{ij} = p_i \cdot p_j$ up to momentum conservation?

$\Updownarrow$ $\qquad$ $\Updownarrow$

$$\theta_{ij}^2 = 2n_i \cdot n_j$$

$$0 = \sum_{i=1}^{M} p_i^\mu = \mathcal{I}^\mu =$$

A: Same as the number of non-isomorphic multigraphs with no leaves (vertices of valency one)

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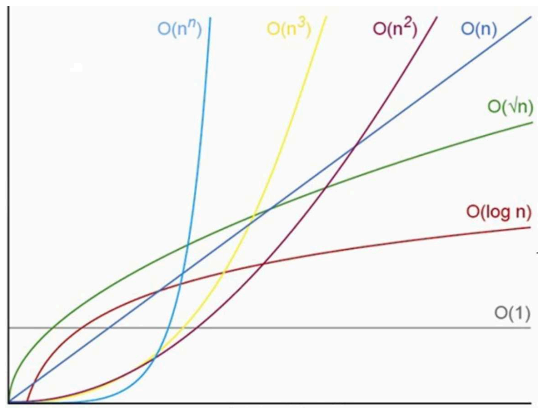
New OEIS Entries!
A307317, A307316

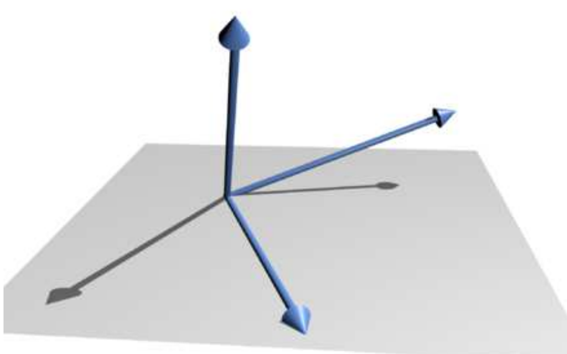
[PTK, Metodiev, Thaler, *to appear soon*]

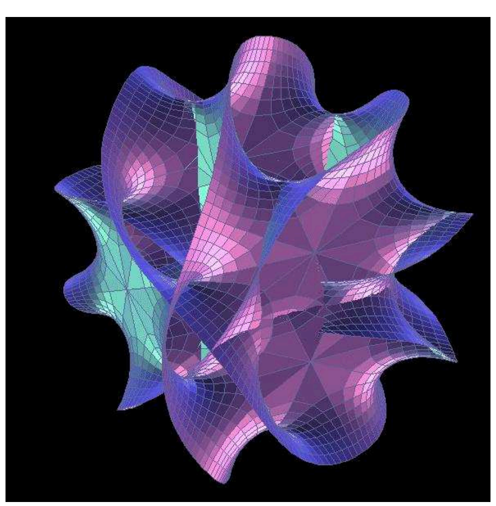
	Leafless Multigraphs	
	Connected	All
Edges d	A307317	A307316
1	0	0
2	1	1
3	2	2
4	4	5
5	9	11
6	26	34
7	68	87
8	217	279
9	**718**	**897**
10	**2 553**	**3 129**
11	**9 574**	**11 458**
12	**38 005**	**44 576**
13	**157 306**	**181 071**
14	**679 682**	**770 237**
15	**3 047 699**	**3 407 332**
16	**14 150 278**	**15 641 159**

Bolded values previously unknown

Summary







Experiment

Computational Complexity

Multiparticle correlators are $\mathcal{O}(M^N)$ to compute in general

$\beta = 2$ EFPs can be computed in $\mathcal{O}(M)$

Why not use $D_2^{(\beta=2)}$? Performance in backup

Linear Tensor Identities

Multiparticle correlators exhibit mysterious linear redundancies

All redundancies understood via cutting graphs and applying master antisymmetrization identity

Counting Superstring Amplitudes

Counting independent kinematic polynomials difficult

Immediate enumeration through multigraphs and new OEIS sequences!

Theory

Rewriting General EFP as Contraction of EFMs

$$\mathrm{EFP}_G = \sum_{i_1=1}^{M} \cdots \sum_{i_N=1}^{M} z_{i_1} \cdots z_{i_N} \prod_{(k,\ell)\in G} 2\eta_{\mu\nu} n_{i_k}^{\mu} n_{i_\ell}^{\nu}$$

$$= \left(\prod_{j=1}^{N} \sum_{i_j=1}^{M} z_{i_j} n_{i_j}^{\mu_1^j} n_{i_j}^{\mu_2^j} \cdots n_{i_j}^{\mu_{v_j}^j} \right) \prod_{(k,\ell)\in G} 2\eta_{\mu^k_{A_{k\ell}} \mu^\ell_{A_{\ell k}}}$$

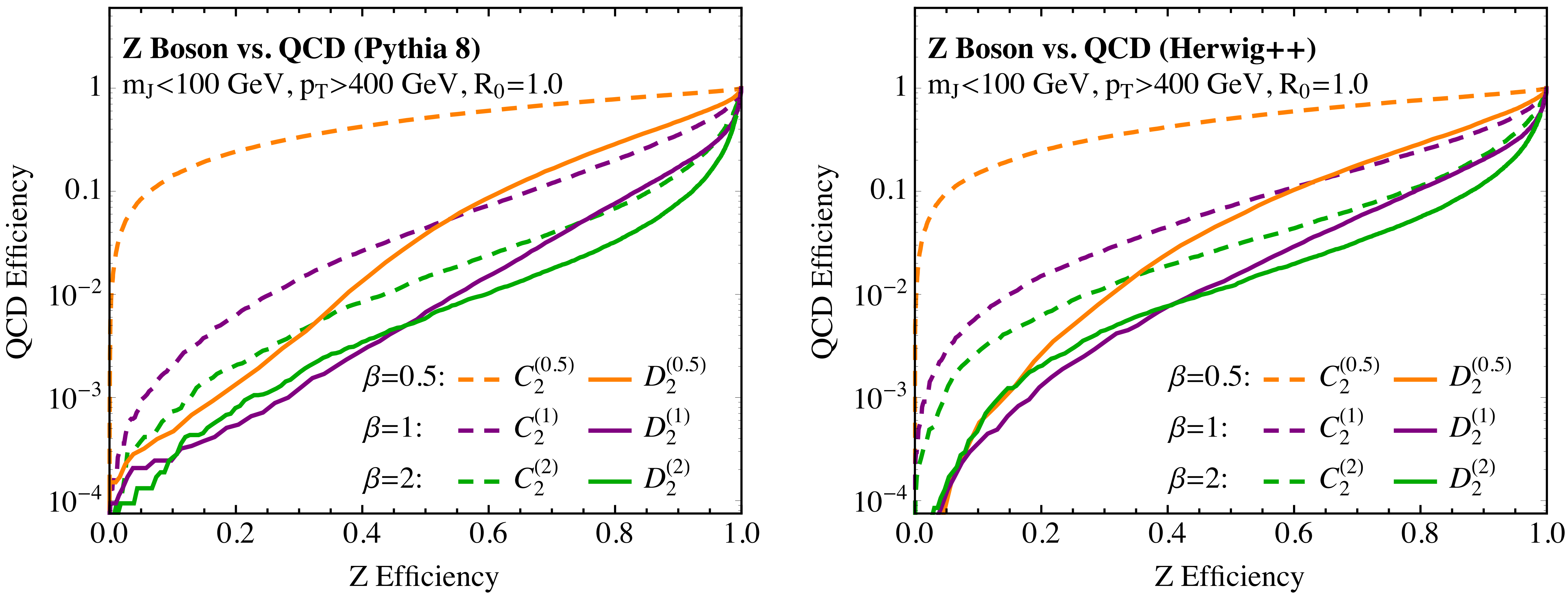
$$= \underbrace{\left(\prod_{j=1}^{N} \mathcal{I}^{\mu_1^j \mu_2^j \cdots \mu_{v_j}^j} \right)}_{\text{EFMs}} \underbrace{\prod_{(k,\ell)\in G} 2\eta_{\mu^k_{A_{k\ell}} \mu^\ell_{A_{\ell k}}}}_{\text{Contraction of edges}}$$

TL;DR – edges of EFP with $\beta = 2$ can be cut and rearranged into EFMs

Two-Prong Classification with Varying β

$\beta = 2$ for both D_2 and C_2 for both Pythia 8 and Herwig++
works better than $\beta = 1$ for Z vs. QCD

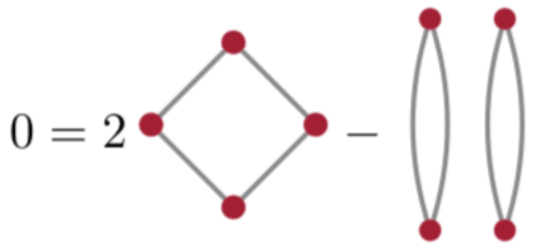
[Larkoski, Moult, Neill, 1409.6298]



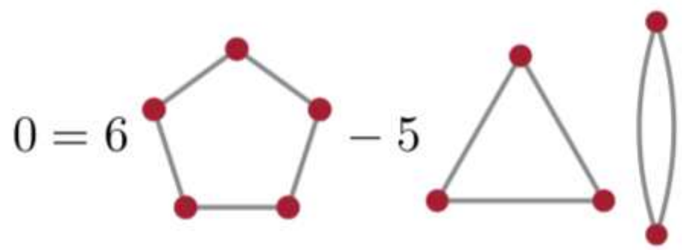
Additional Linear Identities

All identities fundamentally due to antisymmetrizing over more indices than dimensions

Finite spacetime dimension – $\beta = 2$ only



$d \leq 3$



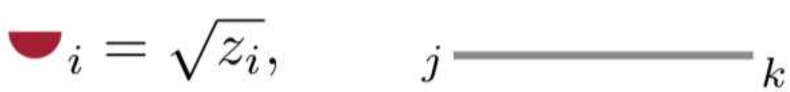
$d \leq 4$

Finite particle number – cutting open vertices

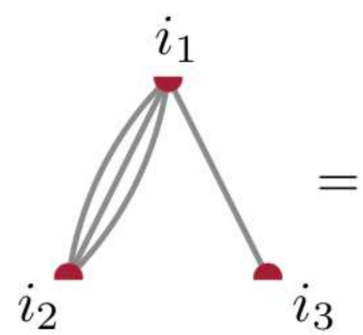
"hamburger tensors"

$$\mathcal{M}_G = \sqrt{z_{i_1} \cdots z_{i_N}} \prod_{(k,\ell)\in G} \theta_{i_k i_\ell}$$

graph rules



$= \sqrt{z_i}, \qquad = \theta_{jk}$



$$= \sqrt{z_{i_1} z_{i_2} z_{i_3}}\, \theta^3_{i_1 i_2} \theta_{i_1 i_3}$$

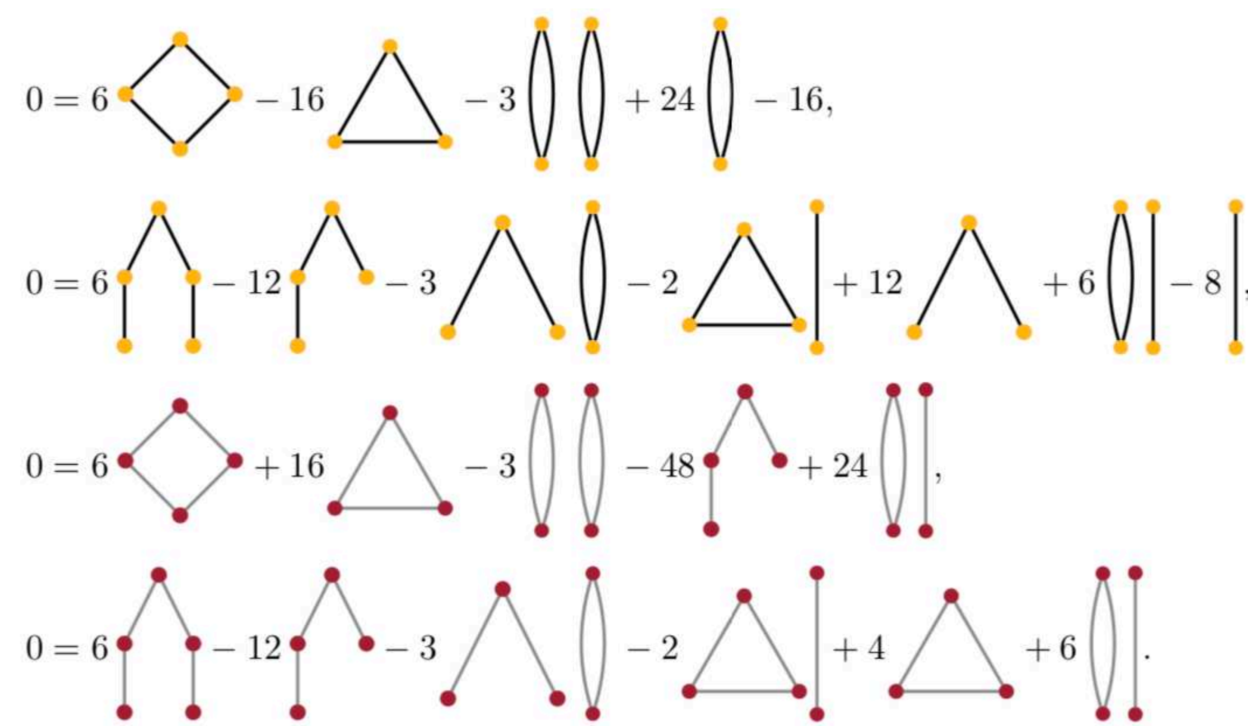
Identities come from antisymmetrizing over $M+1$ or more vertex indices, works for any θ_{jk}!

Euclidean subslicing – e^+e^- only

$$e^+e^- : \quad n_i^\mu = (1, \hat{n})^\mu$$

Presence of 1 means that d dim. tensors are exactly related to $d-1$ dim tensors and hence satisfy more identities

ex. – holds in $d \leq 4$ for e^+e^-



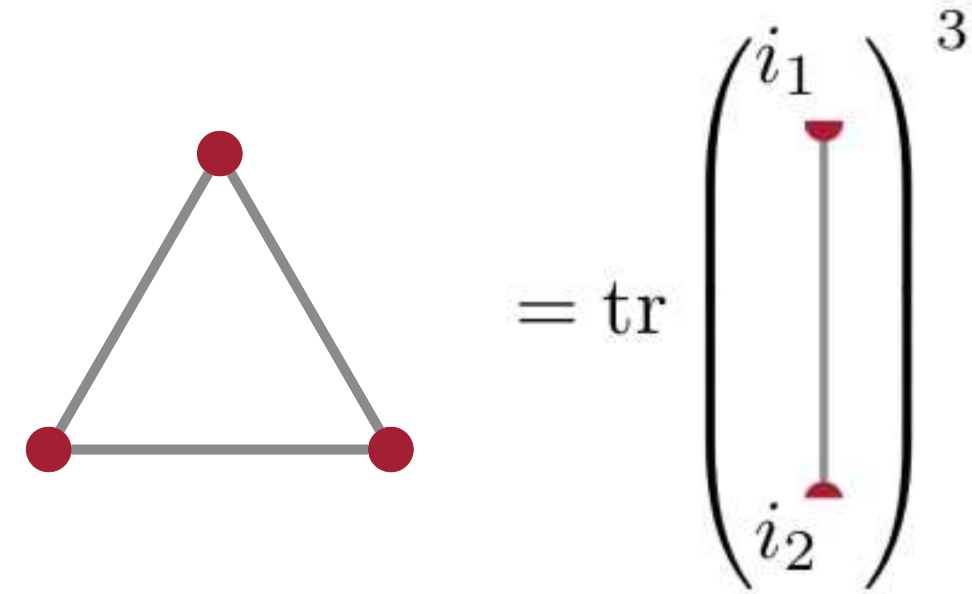
Additional Linear Relation Material

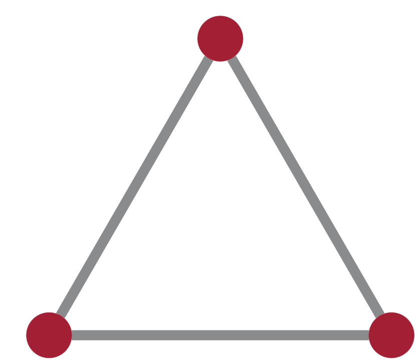
Lorentzian graphs

Euclidean graphs

	$d=0$	$d=1$	$d=2$			$d=3$								…
	1													
	2	−1												
	4	−4	1											
	4	−4		1										
	4	−4			1									
	8	−12	6			−1								
	8	−12		6			−1							
	8	−12	2	4				−1						
	8	−12		6					−1					
	8	−12		4	2					−1				
	8	−12	2		4						−1			
	8	−12		2	4							−1		
	8	−12			6								−1	
	16	−32	24			−8								1
	16	−32	8	16				−8						
	16	−32	4	20			−4	−4						
	16	−32	12	12		−2		−6						
	16	−32		16	8					−8				
	16	−32		20	4		−2		−2	−4				
	16	−32	4	16	4			−4		−4				
	16	−32	4	12	8			−2		−4	−2			
	16	−32	4	20				−4	−4					
	16	−32		24					−8					
	16	−32		16	8				−2	−4		−2		
	16	−32		12	12					−4		−4		
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱

Cutting vertices demonstrates matrix multiplication can be used to calculate some graphs





$\sim \mathcal{O}(M^{2.3728639})$

Works for all angular measures