Probing QCD with Energy Flow Observables

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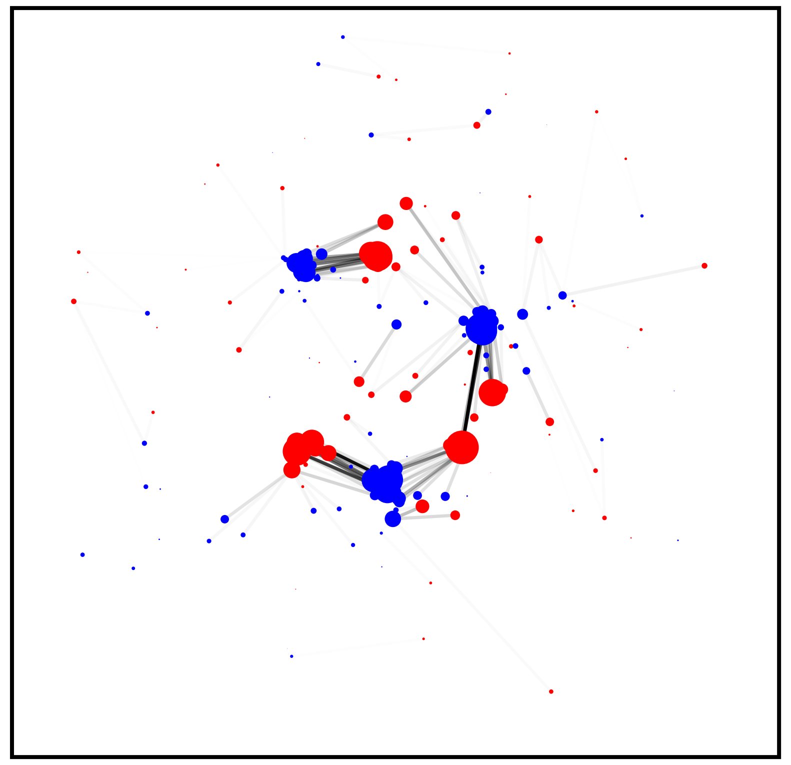
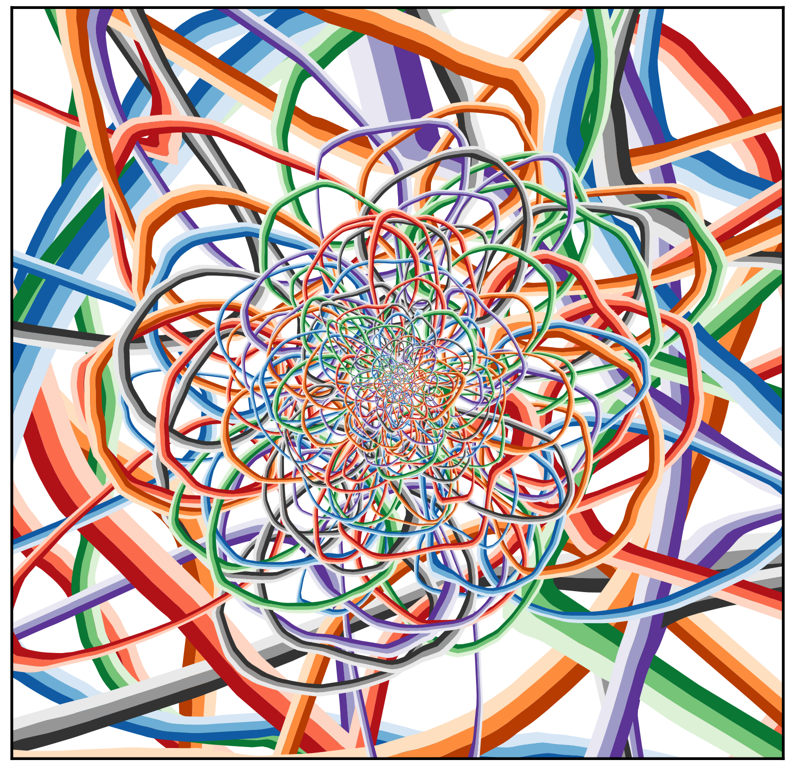
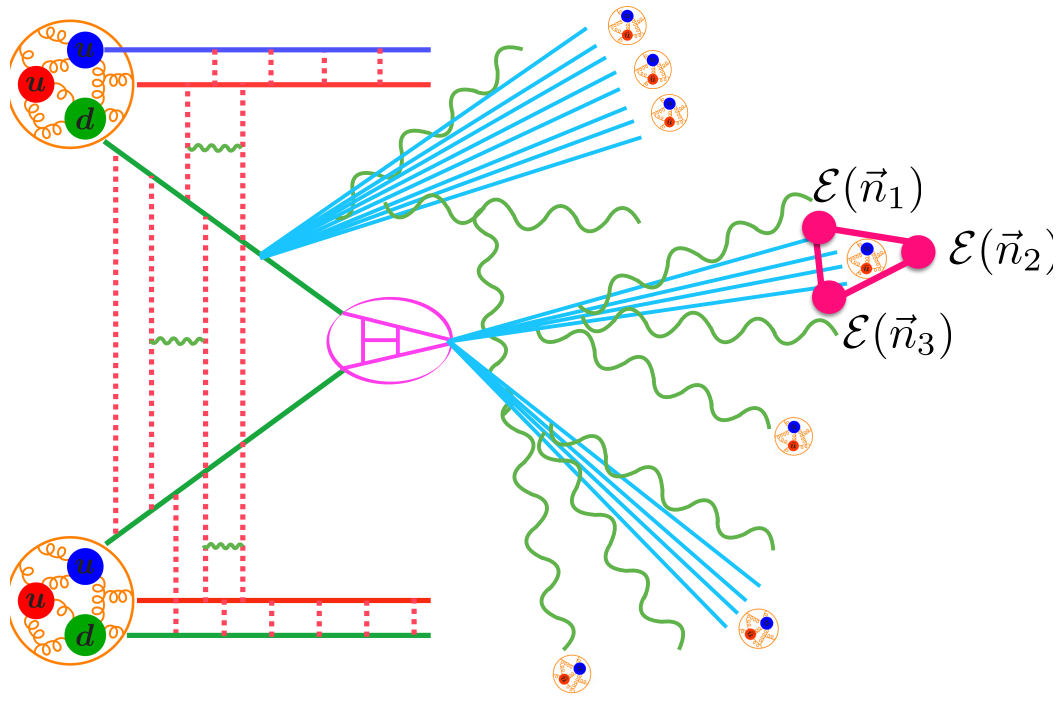
CEPC Workshop

October 27, 2020



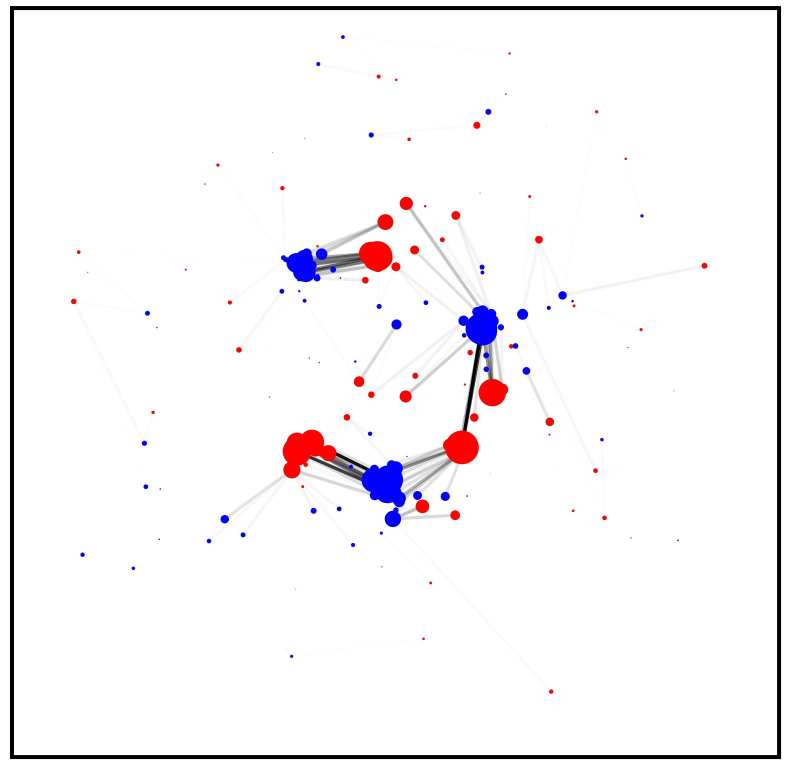




Collider Event Fundamentals

Energy Flow Observables

Energy-Energy Correlators



Collider Event Fundamentals

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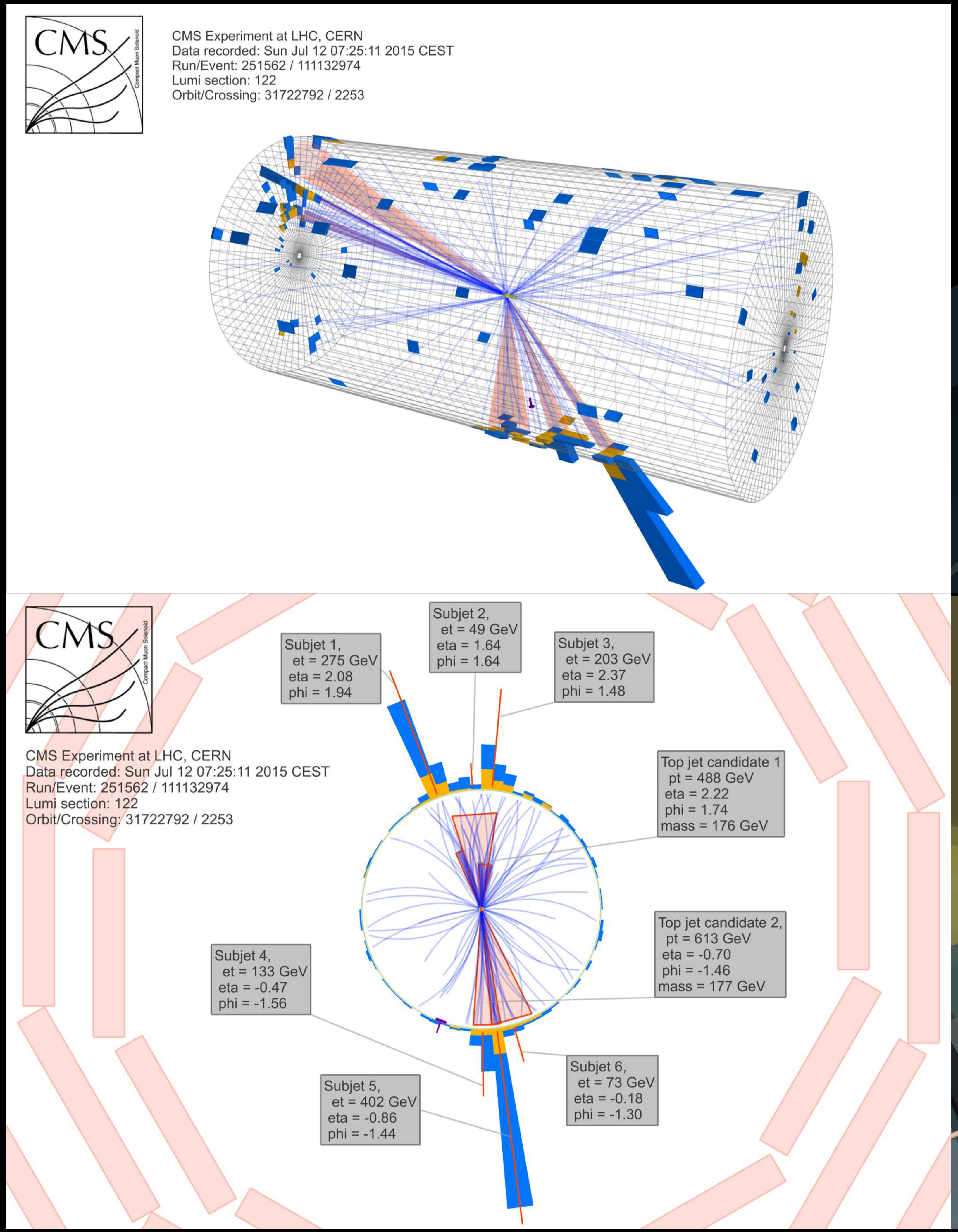
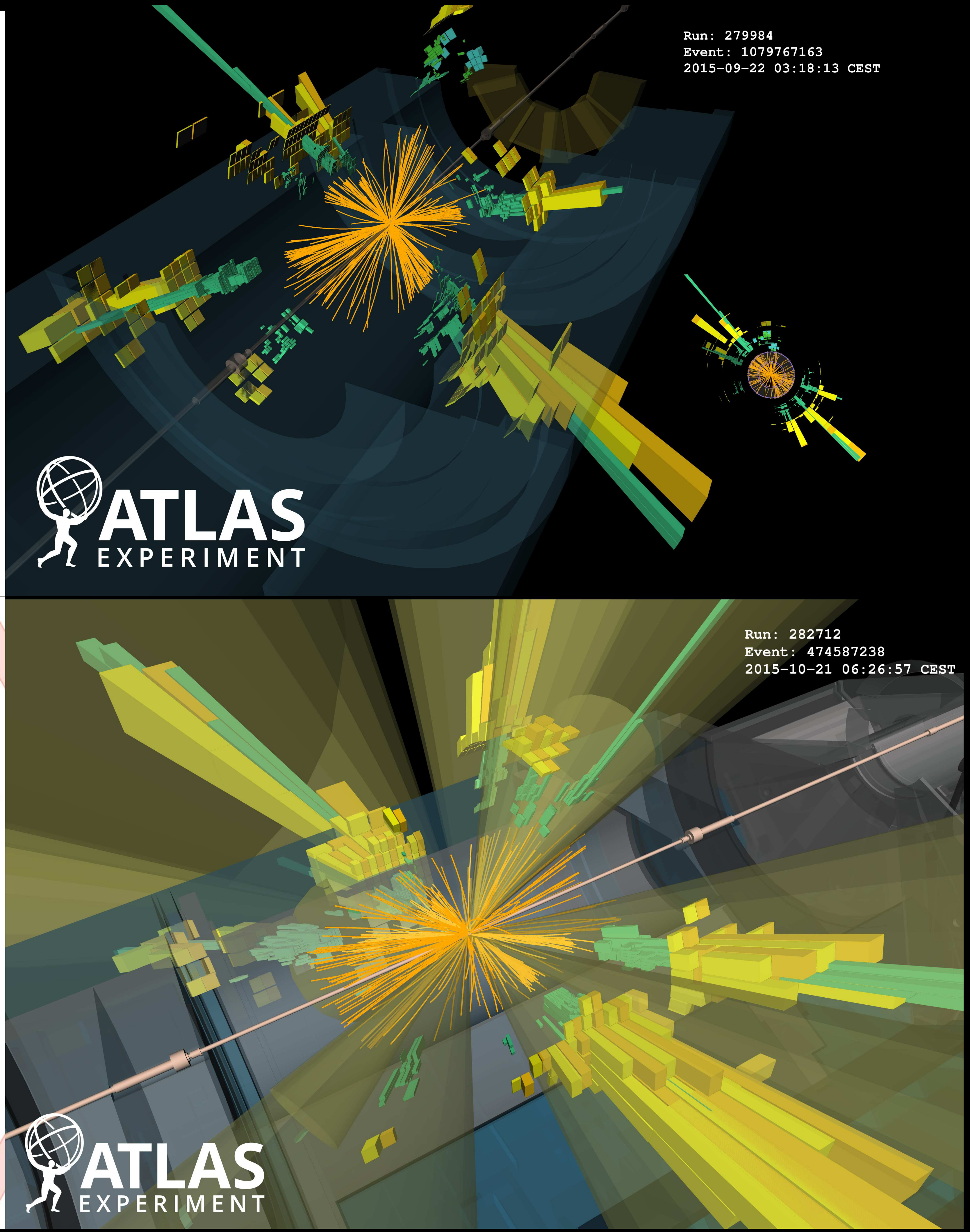
Energy-Energy Correlators

Hadronic Events at the Large Hadron Collider

Jets (collimated sprays of color-neutral particles) are ubiquitous at high-energy colliders



CMS hadronic $t\bar{t}$ event



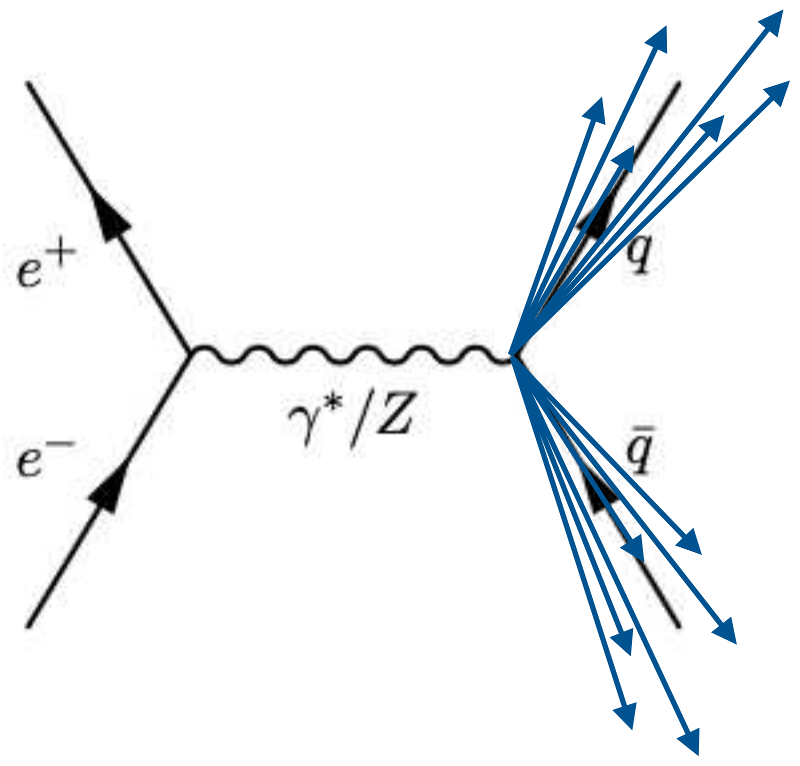
ATLAS high jet multiplicity events

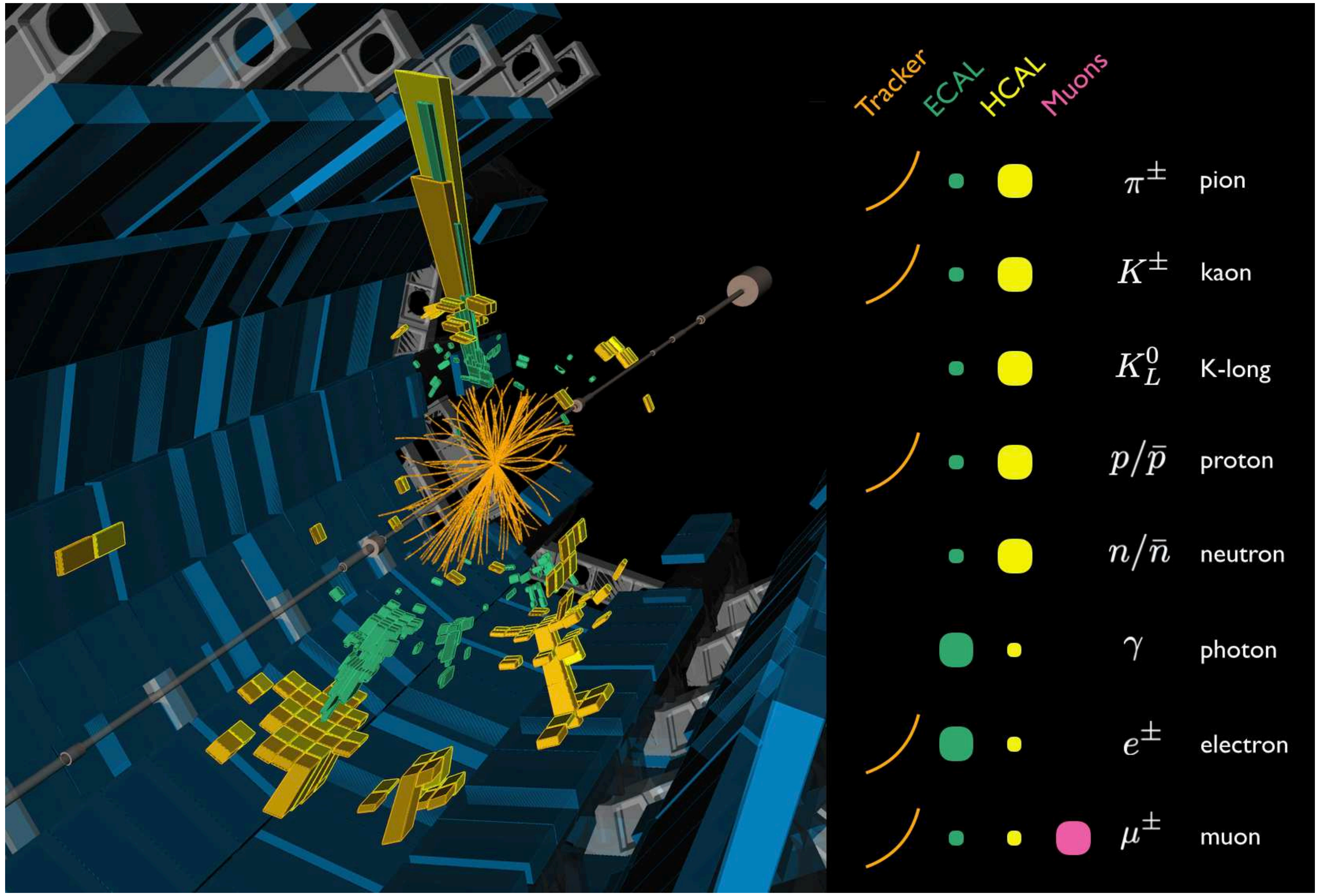
Hadronic Events as a Particle Point Cloud

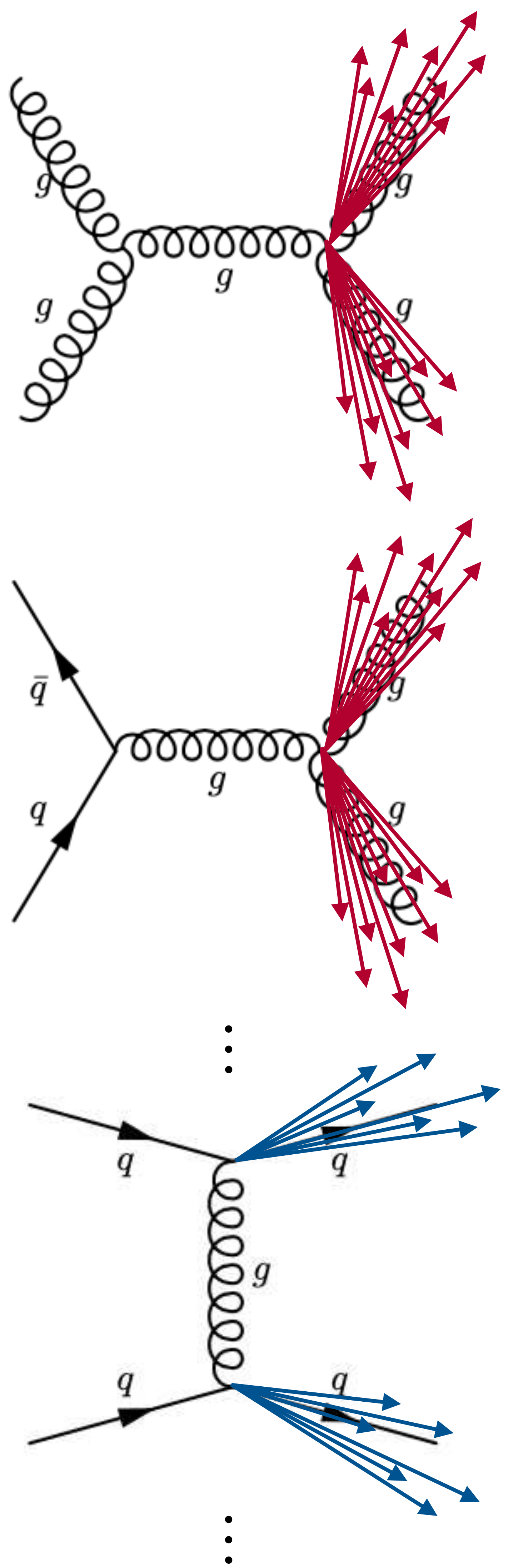
Jets in pp

High-energy collisions produce final state particles with
energy, *direction*, *charge*, *flavor*, and *other quantum numbers*

Jets in e^+e^-







Jet Formation in Theory

Hard collision

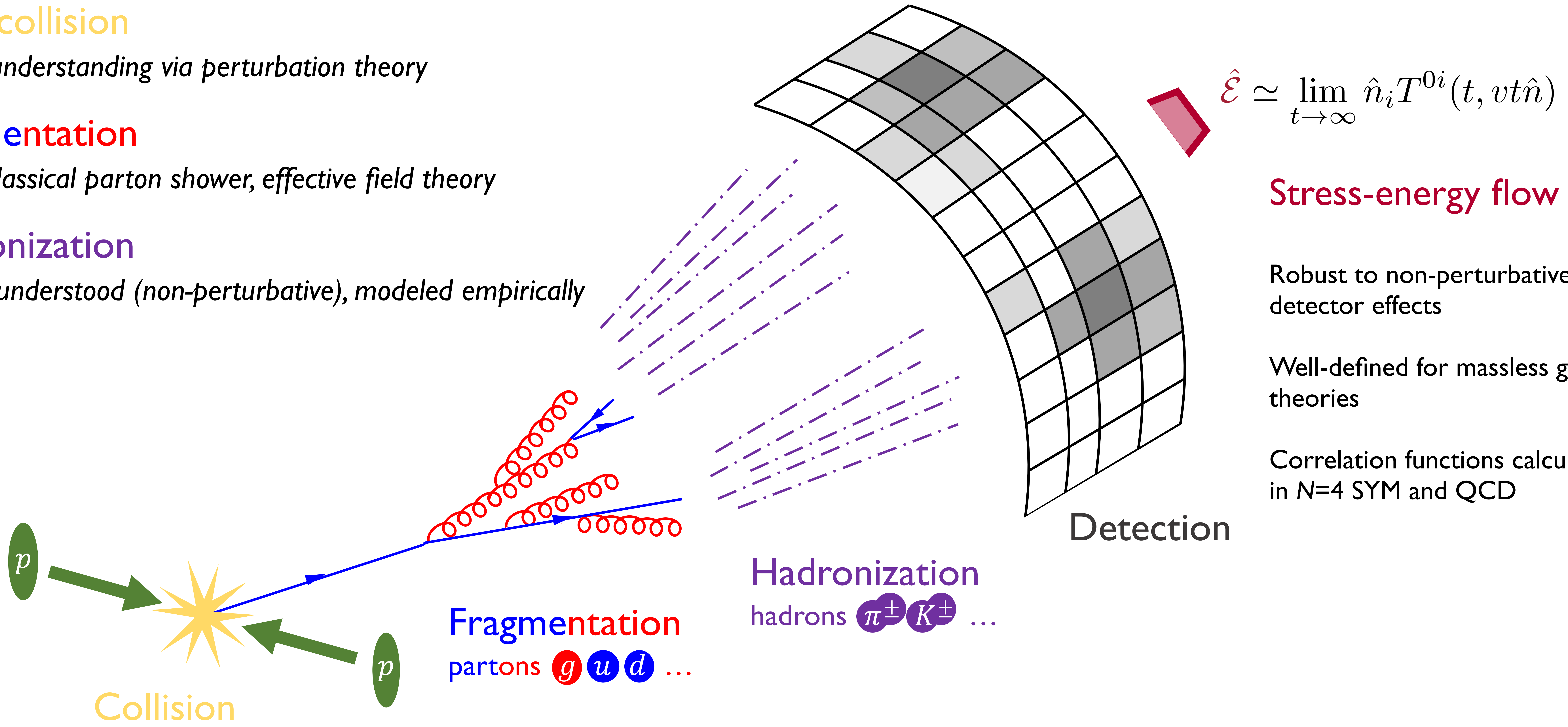
Good understanding via perturbation theory

Fragmentation

Semi-classical parton shower, effective field theory

Hadronization

Poorly understood (non-perturbative), modeled empirically



Stress-energy flow

Robust to non-perturbative and detector effects

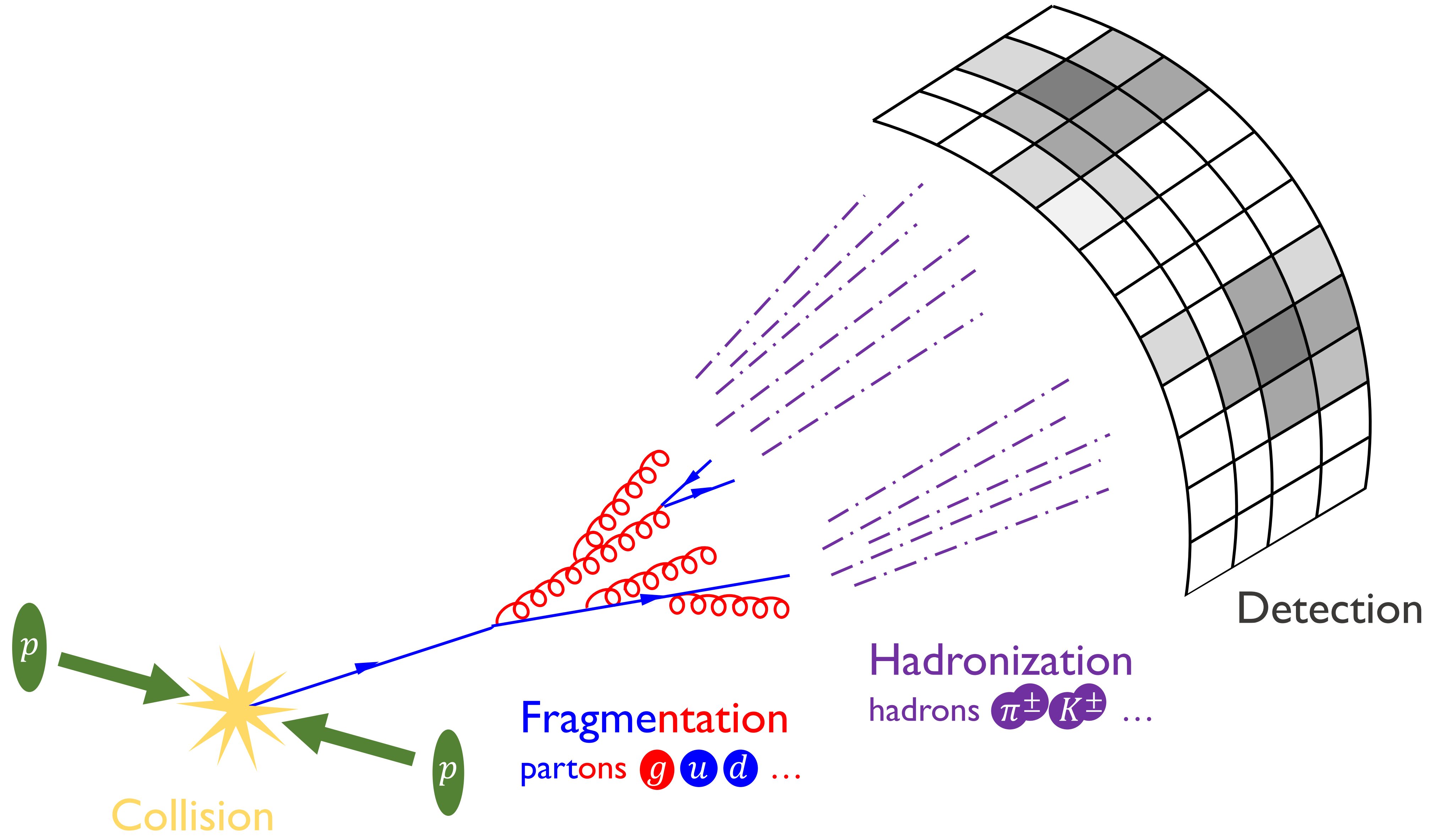
Well-defined for massless gauge theories

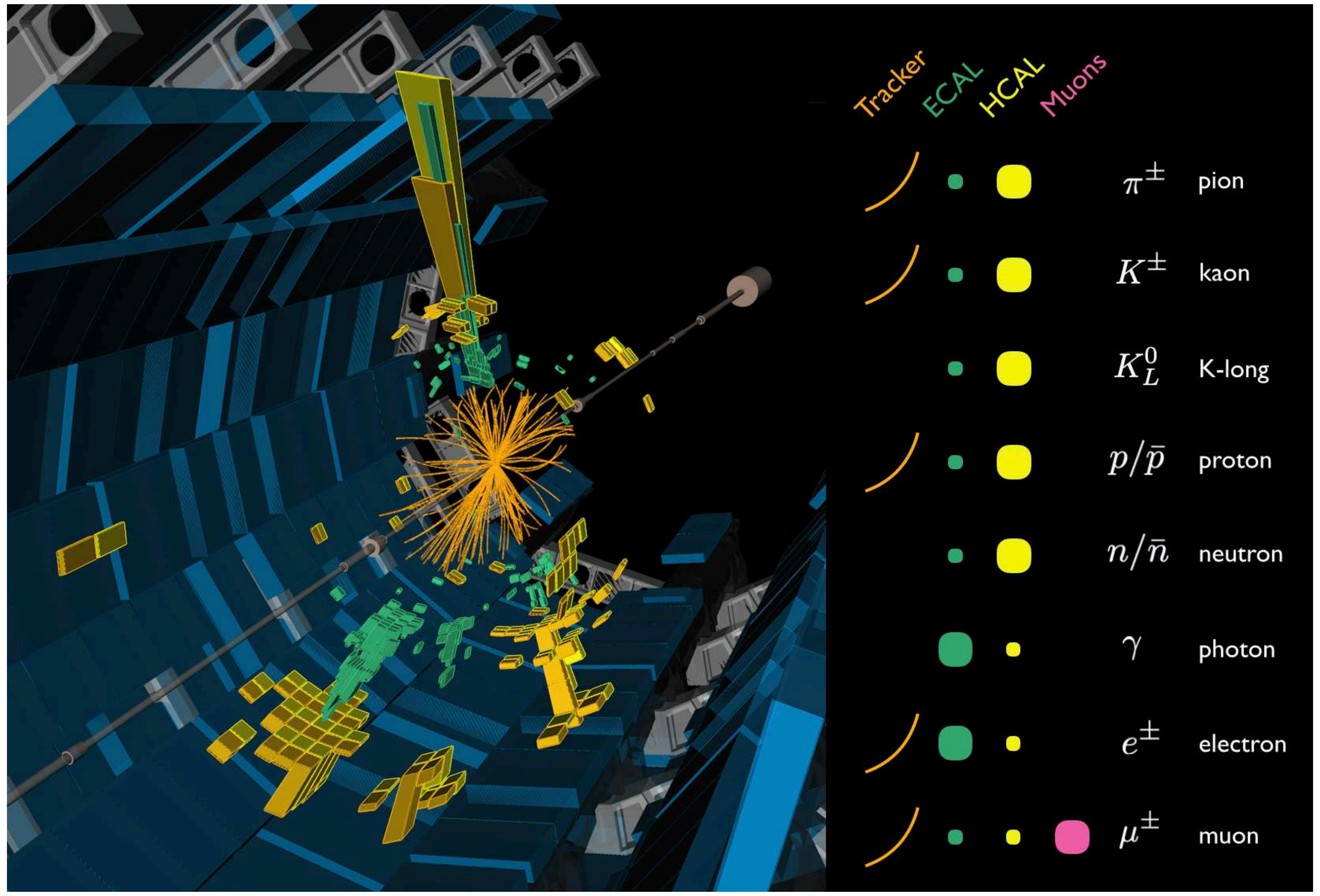
Correlation functions calculated in N=4 SYM and QCD

Diagram by Eric Metodiev

[Sveshnikov, Tkachov, PLB 1996; Hofman, Maldacena, JHEP 2008; Mateu, Stewart, Thaler, PRD 2013; Belitsky, Hohenegger, Korchemsky, Sokatchev, Zhiboedov, PRL 2014; Chen, Moult, Zhang, Zhu, 2004.11381; PTK, Moult, Thaler, Zhu, *to appear soon*]

Events in Theory vs. Experiment





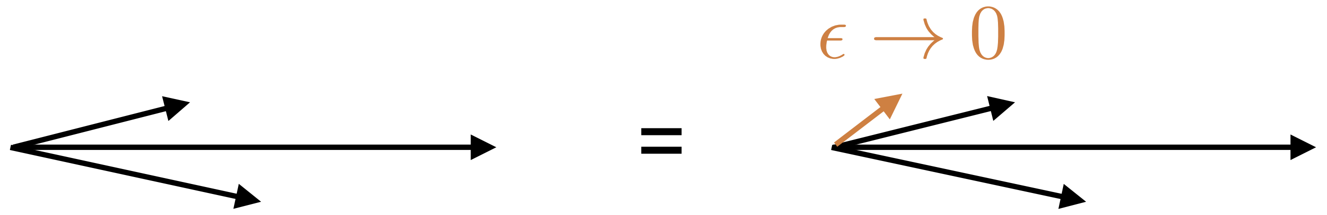
What information is both theoretically and experimentally robust?

Theoretically and Experimentally Robust Information

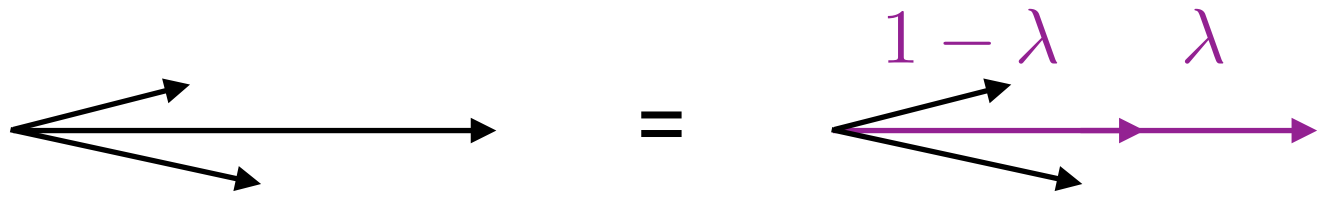
***Infrared* and *Collinear* Safe Information**

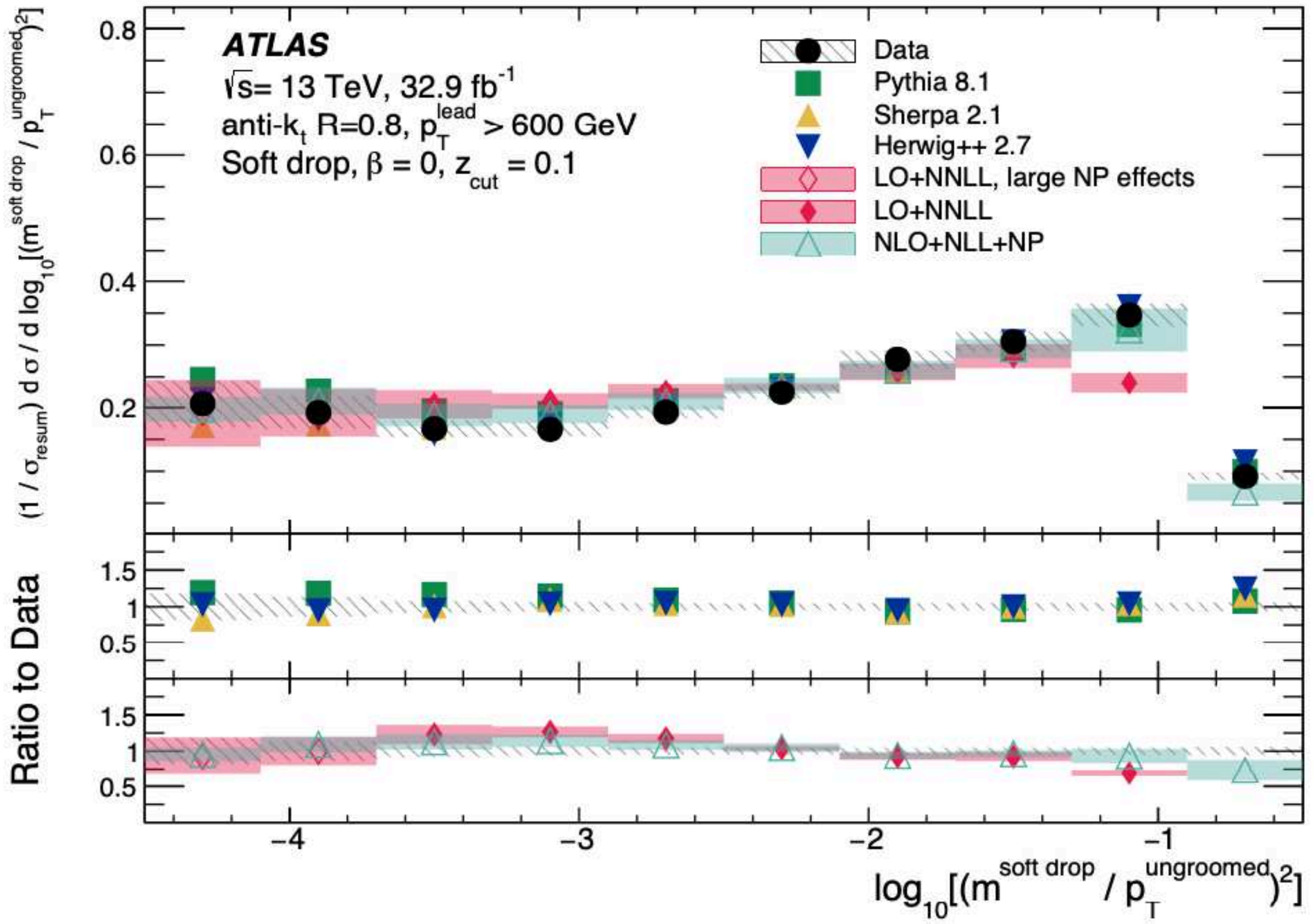
e.g. groomed jet substructure at the LHC

Infrared (IR) safety – observable is unchanged under addition of a soft particle



Collinear (C) safety – observable is unchanged under a collinear splitting of a particle





[ATLAS, PRL 2018]

Theoretically

QCD has soft and collinear divergences associated with gluon radiation

$$dP_{i\to ig} \simeq \frac{2\alpha_s}{\pi} C_a \frac{d\theta}{\theta} \frac{dz}{z}$$

$$C_q = C_F = 4/3$$
$$C_g = C_A = 3$$

Experimentally

IRC safety is a statement of *linearity* in energy and *continuity* in geometry

[See backup for more on IRC safety]

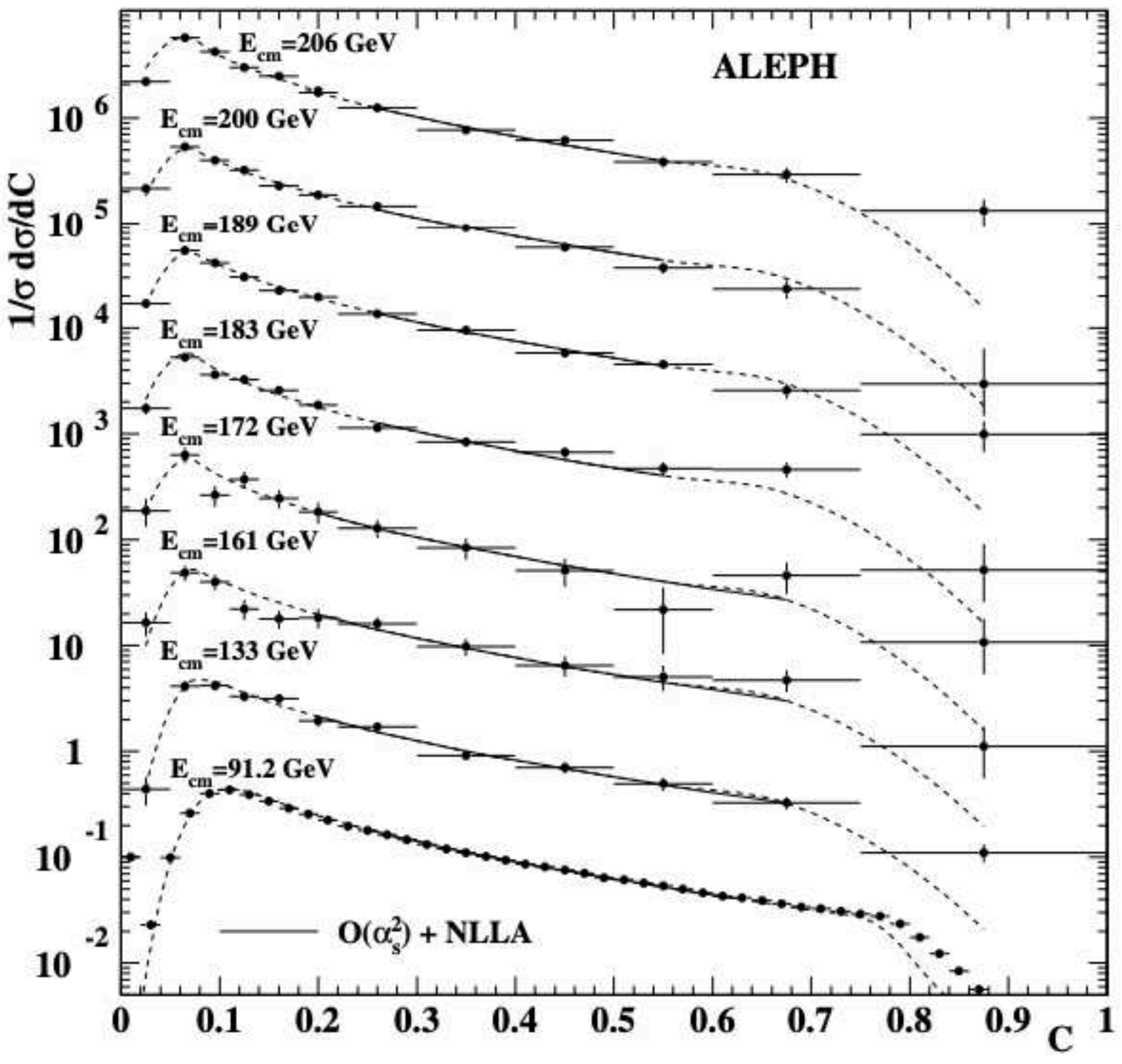
[Sveshnikov, Tkachov, hep-ph/9512370; Tkachov, hep-ph/9601308; Cherzor, Sveshnikov, hep-ph/9710349; PTK, Metodiev, Thaler, JHEP 2020]

Event Shapes Past and Present

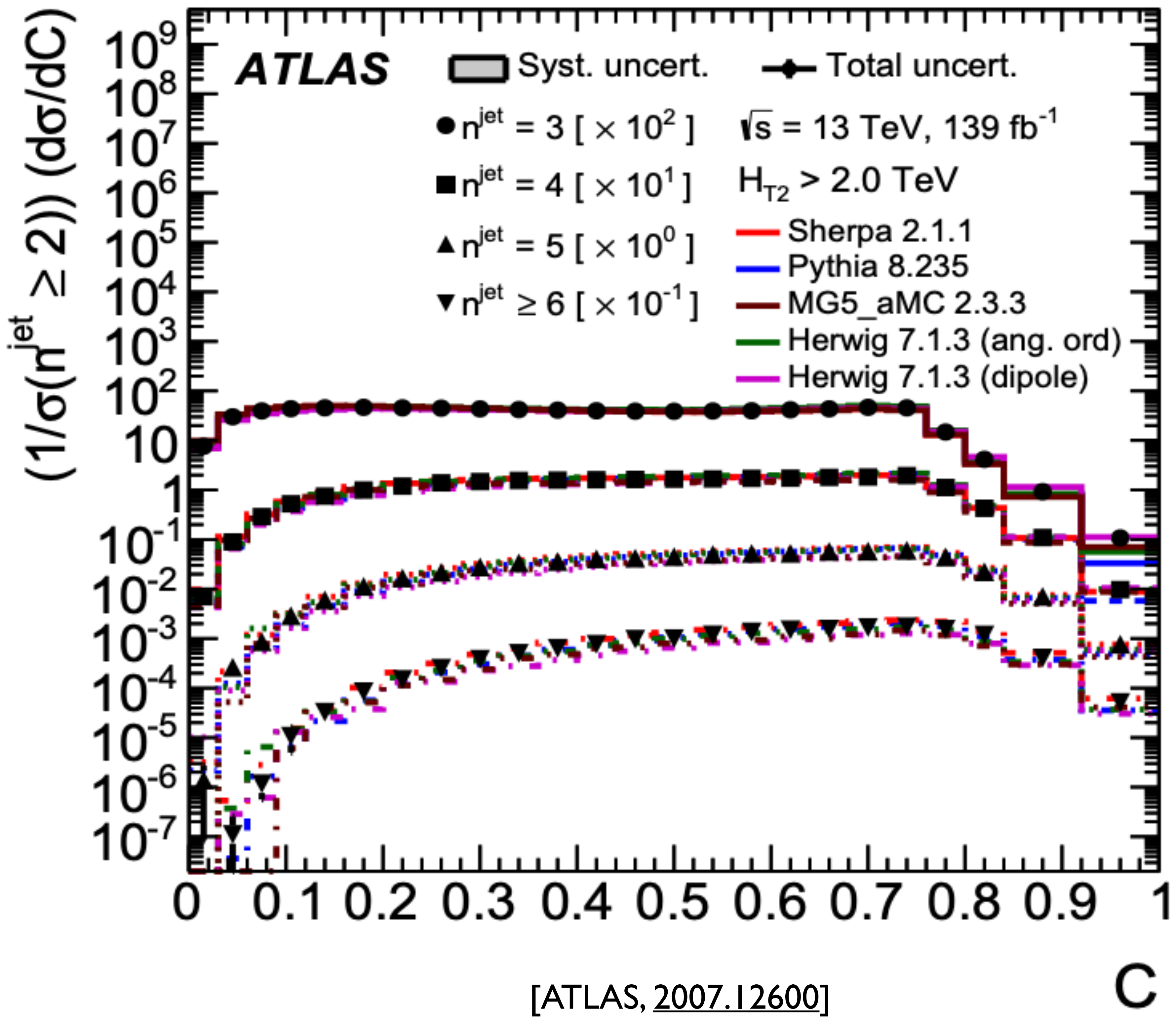
Ubiquitous collider observables that probe QCD radiation

e.g. Sphericity tensor: $\Theta^{ij} = \sum_k z_k \hat{n}_k^i \hat{n}_k^j, \quad z_k = \frac{|\vec{p}_k|}{\sum_i |\vec{p}_i|}, \quad \hat{n}_k^i = \frac{(\vec{p}_k)^i}{|\vec{p}_k|}$ $\quad C = 3(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3)$ $\quad D = 27(\lambda_1\lambda_2\lambda_3)$

Eigenvalues of Θ^{ij}



[ALEPH, EPJC 2004]



[ATLAS, 2007.12600]

Have been used to extra α_s by comparing to precision QCD calculations

Master Formula for Collider Physics

$$\sigma_{\text{obs}} \sim \frac{1}{2E_{\text{CM}}^2} \underbrace{\sum_{n=2}^{\infty} \int d\Phi_n}_{\text{phase space}} \underbrace{|\mathcal{M}_{AB\to 12\cdots n}|^2}_{\text{amplitude}} \underbrace{f_{\text{obs}}(\Phi_n)}_{\text{observable}}$$

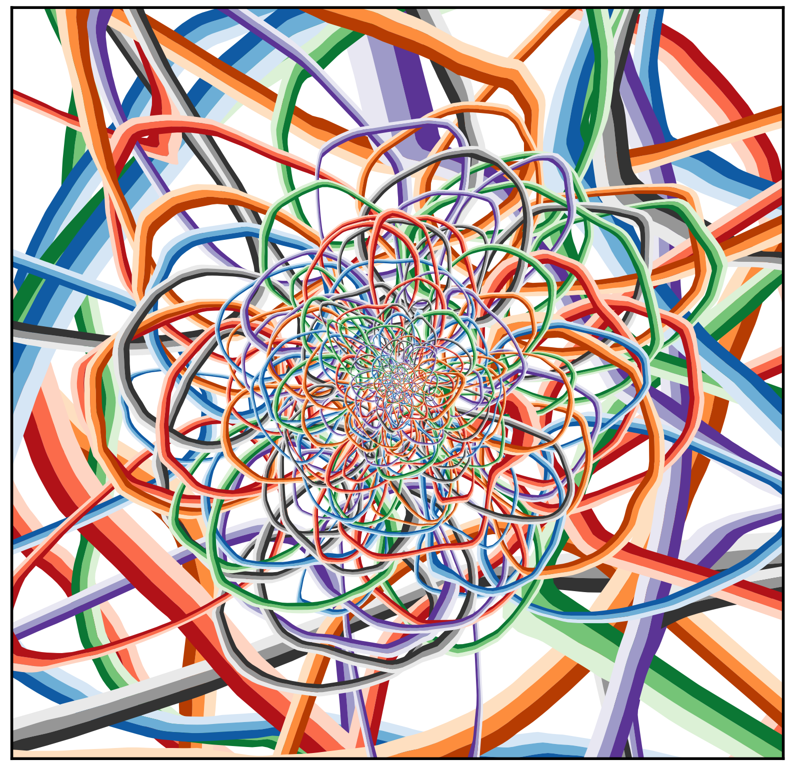
Experiment

Fixed-order calculations

Resummed calculations

Guiding question for rest of talk:

Can we develop a systematic understanding of observables to inform selection of quantities to measure/calculate?



Collider Event Fundamentals

Energy Flow Observables

Energy-Energy Correlators

Multiparticle Correlators

Expanded upon during the jet substructure revolution at the LHC

Definition of energy factor and pairwise angular

$pp: \quad z_i = \frac{p_{Ti}}{\sum_j p_{Tj}} \qquad \theta_{ij}^2 = 2 n_i^\mu n_{j\mu} = 2\frac{p_i^\mu}{p_{Ti}}\frac{p_{\mu j}}{p_{Tj}} \simeq (y_i - y_j)^2 + (\phi_i - \phi_j)^2$

$e^+e^-: \quad z_i = \frac{E_i}{\sum_j E_j} \qquad \theta_{ij}^2 = 2 n_i^\mu n_{j\mu} = 2\frac{p_i^\mu}{E_i}\frac{p_{j\mu}}{E_j}$

Multiplicity

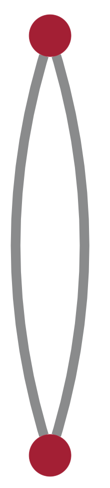
$$\sum_{i=1}^{M} 1$$

- Track multiplicity
- Subjet multiplicity
- Calo multiplicity

Mass

$$\frac{1}{2}\sum_{i=1}^{M}\sum_{j=1}^{M} z_i z_j \theta_{ij}^2$$

- Track mass
- Soft Drop mass
- Trimmed mass

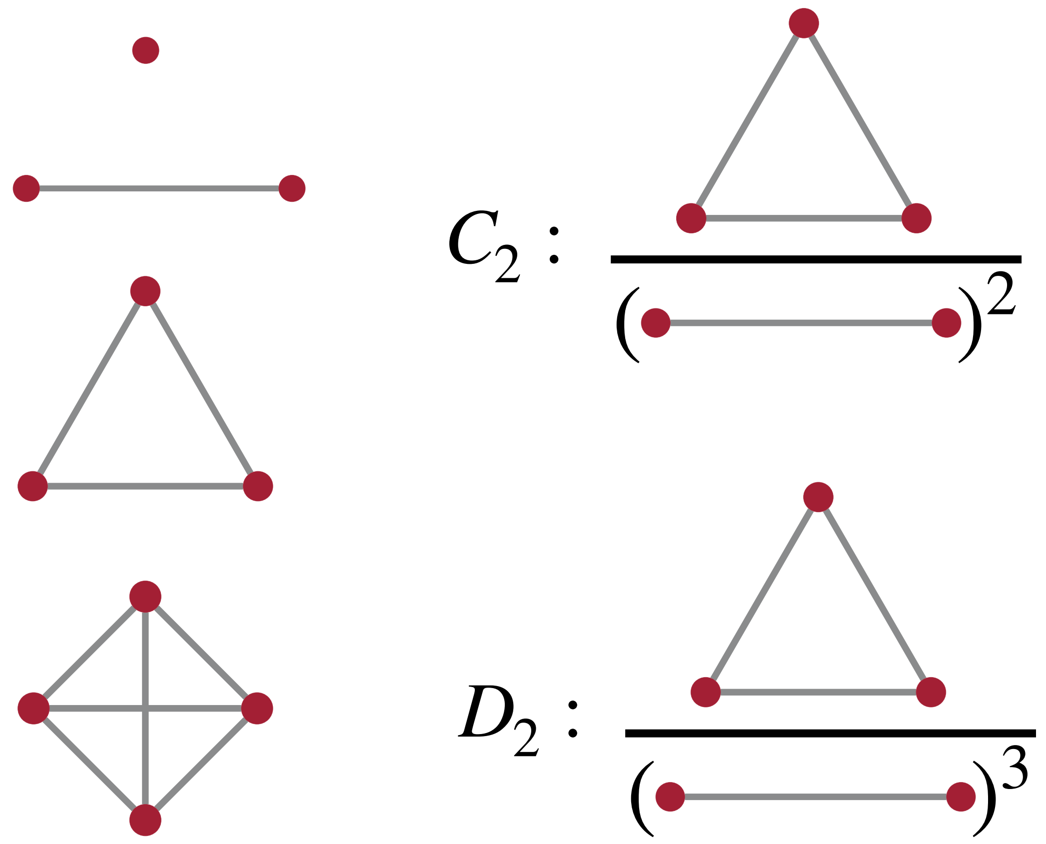
$\frac{1}{2}$ 

Energy Correlation Functions (ECFs)

$$\sum_{i_1=1}^{M}\cdots\sum_{i_N=1}^{M} z_{i_1}\cdots z_{i_N}\prod_{j<k}\theta_{i_j i_k}^{\beta}$$

[Larkoski, Salam, Thaler, 1305.0007]
[Larkoski, Moult, Neill, 1409.6298]

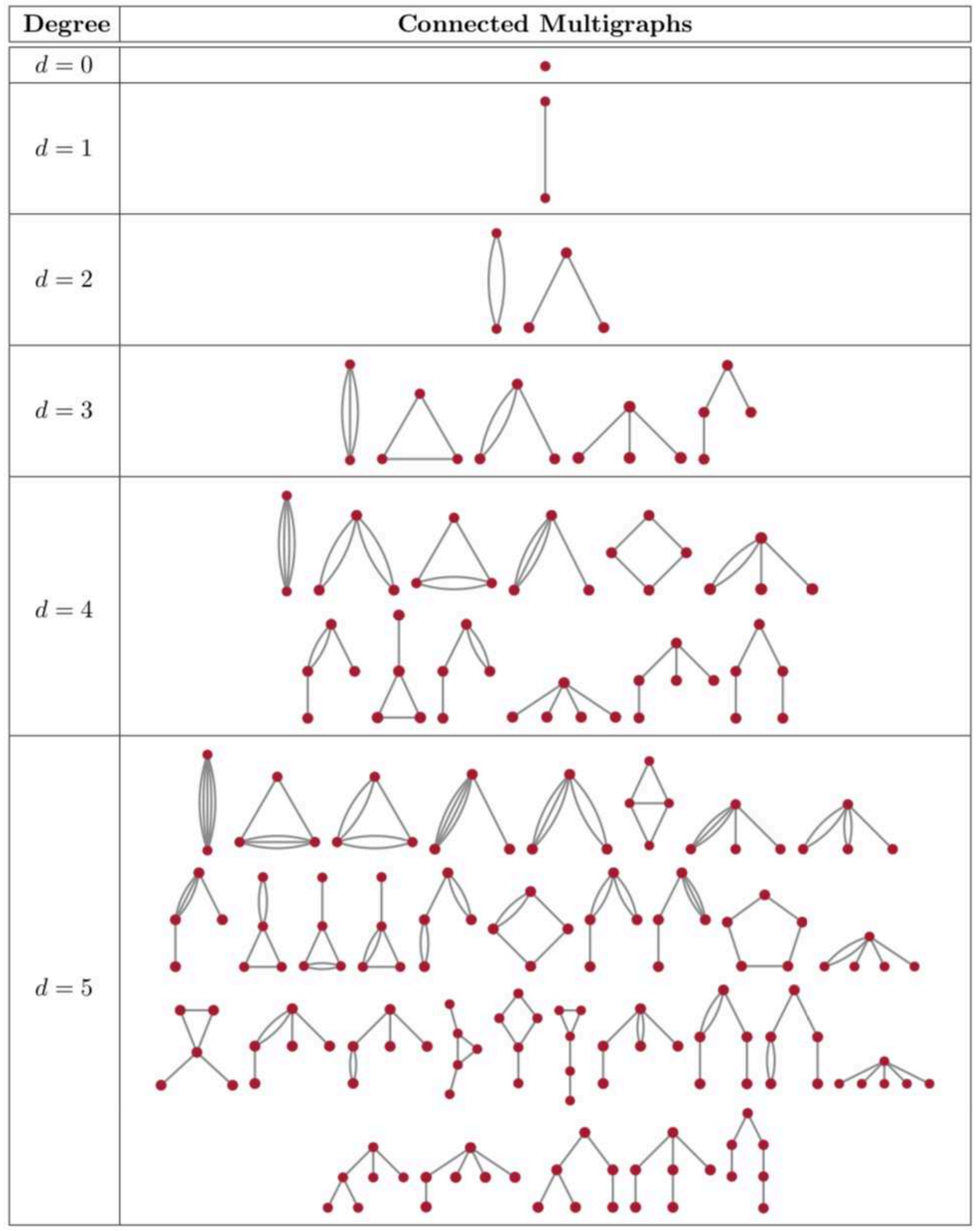
- Ratios typical, D_2, C_2, C_3, etc.
- Used for multi-prong tagging
- Generalized ECFs also useful (angular part not monomial)



Energy Flow Polynomials (EFPs)

$$\sum_{i_1=1}^{M}\cdots\sum_{i_N=1}^{M} z_{i_1}\cdots z_{i_N}\prod_{(j,k)\in G}\theta_{i_j i_k}^{\beta}$$

[PTK, Metodiev, Thaler, JHEP 2018]



Graphs represent correlators

vertex $\leftrightarrow$ energy factor edge $\leftrightarrow$ pairwise angle

$j \longleftrightarrow \sum_{i_j=1}^{M} z_{i_j} \qquad k \quad l \longleftrightarrow \theta_{i_k i_l}$

Energy Flow Polynomials (EFPs)

Obtained via systematically expanding in energies and angles

[PTK, Metodiev, Thaler, JHEP 2018]

Definition of energy factor and pairwise angular

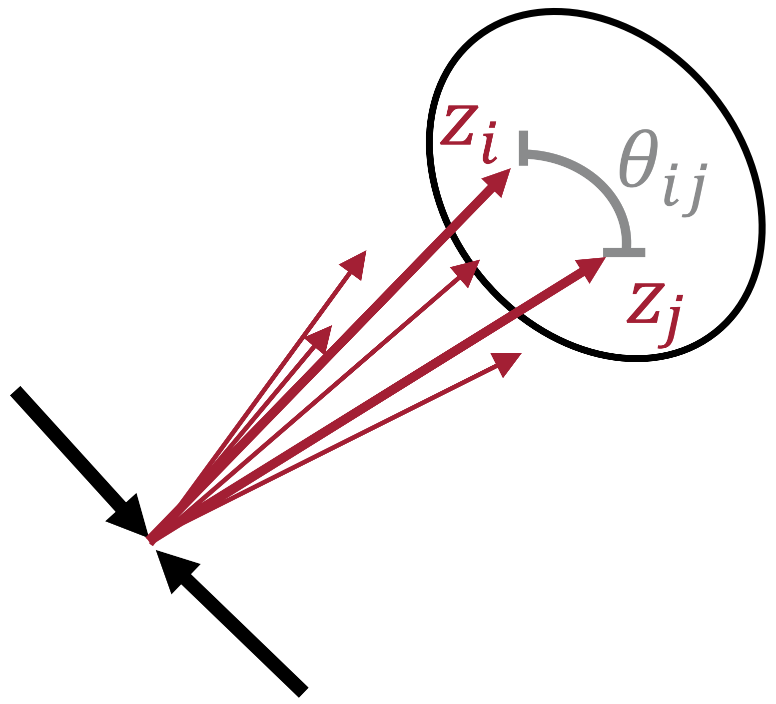
pp : $z_i = \dfrac{p_{Ti}}{\sum_j p_{Tj}}$ $\quad \theta_{ij}^2 = 2 n_i^\mu n_{j\mu} = 2 \dfrac{p_i^\mu}{p_{Ti}} \dfrac{p_{\mu j}}{p_{Tj}} \simeq (y_i - y_j)^2 + (\phi_i - \phi_j)^2$

e^+e^- : $z_i = \dfrac{E_i}{\sum_j E_j}$ $\quad \theta_{ij}^2 = 2 n_i^\mu n_{j\mu} = 2 \dfrac{p_i^\mu}{E_i} \dfrac{p_{j\mu}}{E_j}$

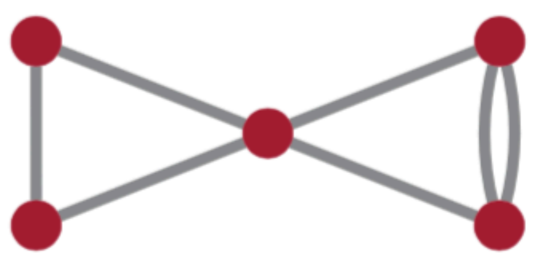
Energy Flow Polynomials (EFPs)

$$\sum_{i_1=1}^{M} \cdots \sum_{i_N=1}^{M} z_{i_1} \cdots z_{i_N} \prod_{(j,k)\in G} \theta_{i_j i_k}^{\beta}$$

[PTK, Metodiev, Thaler, 1712.07124]



e.g.



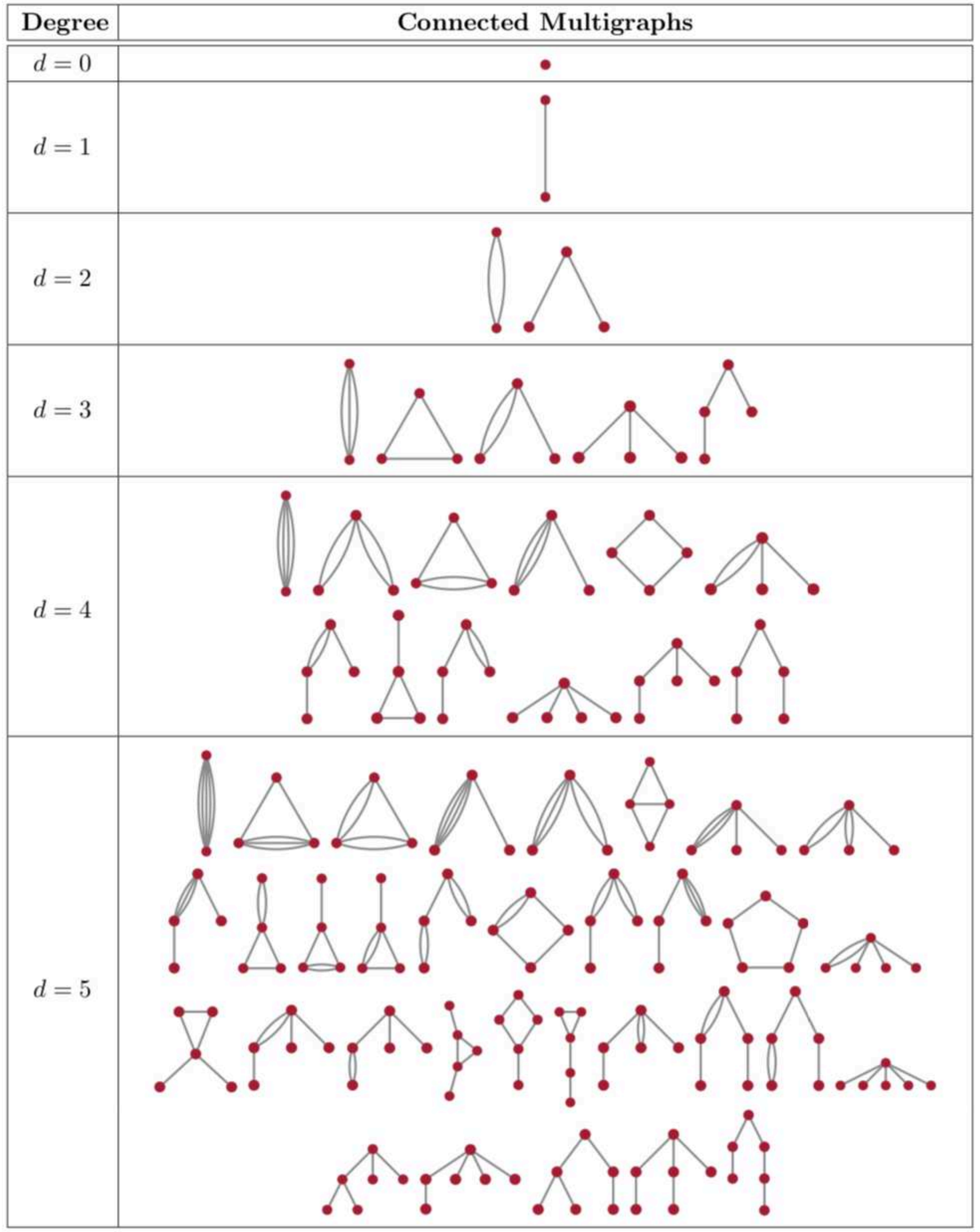
$$= \sum_{i_1=1}^{M} \sum_{i_2=1}^{M} \sum_{i_3=1}^{M} \sum_{i_4=1}^{M} \sum_{i_5=1}^{M} z_{i_1} z_{i_2} z_{i_3} z_{i_4} z_{i_5} \theta_{i_1 i_2} \theta_{i_2 i_3} \theta_{i_1 i_3} \theta_{i_1 i_4} \theta_{i_1 i_5} \theta_{i_4 i_5}^2$$

Graphs represent correlators

vertex ↔ energy factor edge ↔ pairwise angle

$j \longleftrightarrow \sum_{i_j=1}^{M} z_{i_j}$ $\qquad k \;\text{———}\; l \longleftrightarrow \theta_{i_k i_l}$

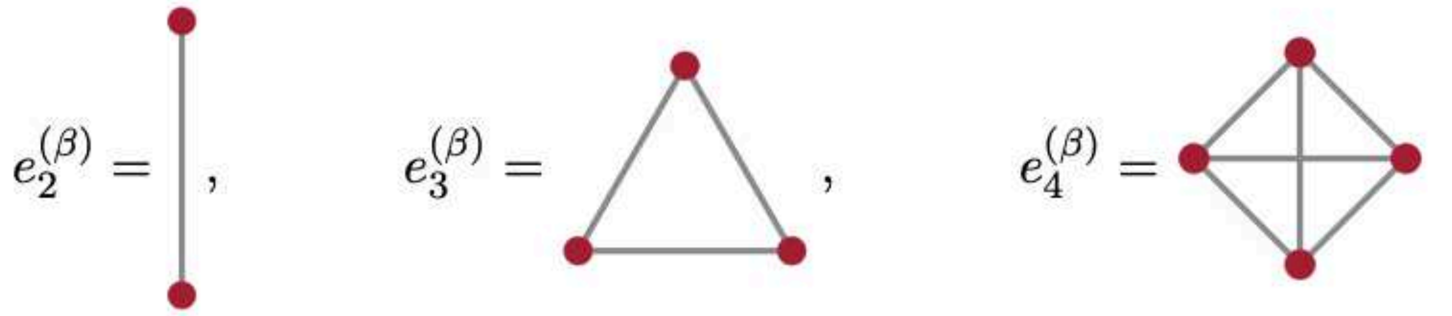
Organized by number of edges *d*



Familiar Observables as EFPs

Energy correlation functions are complete graphs

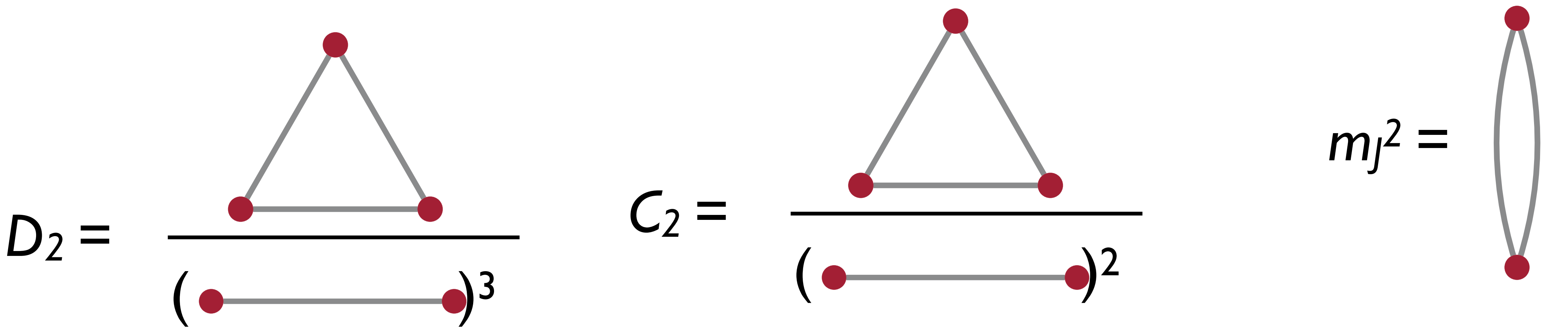
Energy Correlation Functions: $e_N^{(\beta)} = \sum_{i_1}^{M} \cdots \sum_{i_N=1}^{M} z_{i_1} \cdots z_{i_N} \prod_{k<\ell\in\{1,\dots,N\}} \theta_{i_k i_\ell}^{\beta}$



with measure choice β

A.J. Larkoski, G.P. Salam, J. Thaler, 1305.0007

Observables commonly used at LHC

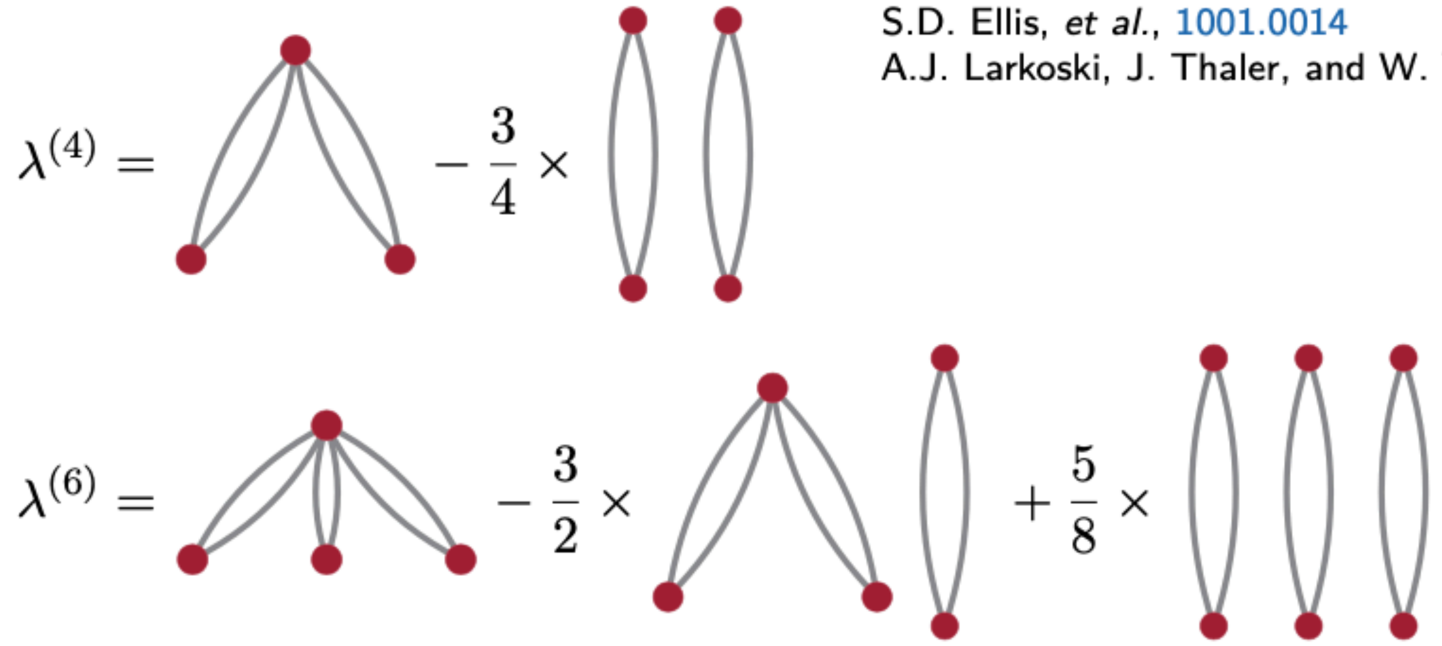


[Larkoski, Moult, Neill, 2014] [Larkoski, Salam, Thaler, 2013]

Even angularities are exact linear combinations of EFPs

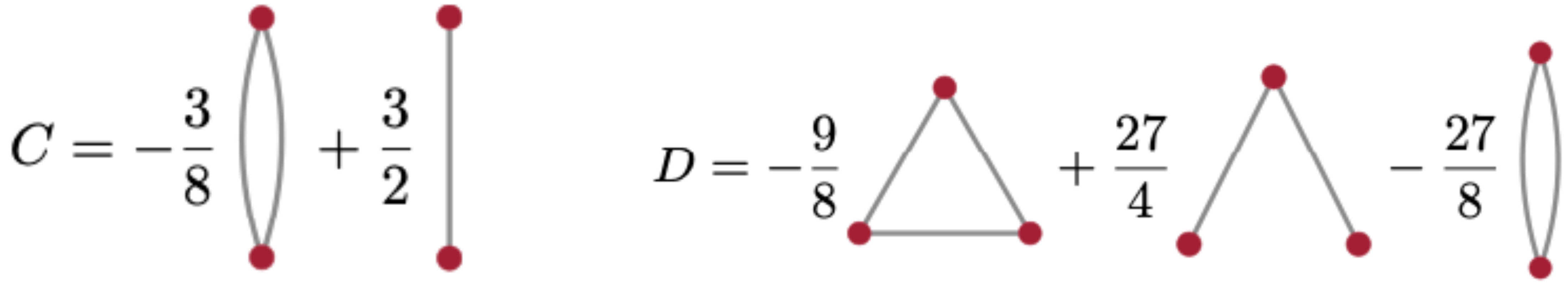
Angularities: $\lambda^{(\alpha)} = \sum_{i=1}^{M} z_i \theta_i^{\alpha}$ using p_T-centroid axis

C.F. Berger, T. Kucs, and G. Sterman, hep-ph/0303051
S.D. Ellis, *et al.*, 1001.0014
A.J. Larkoski, J. Thaler, and W. Waalewijn, 1408.3122



Even classic event shapes!

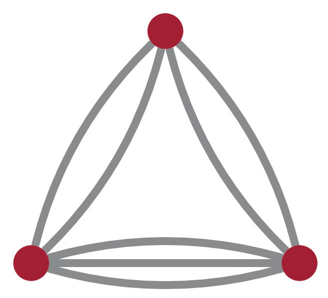
with $\beta = 2$



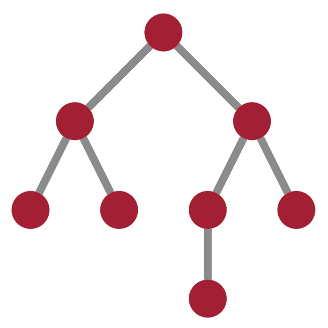
EFPs as a Linear Basis of IRC-Safe Observables

Any IRC-safe observable is a linear combination of EFPs (via the Stone-Weierstrass approximation theorem)!

[See backup for derivation]

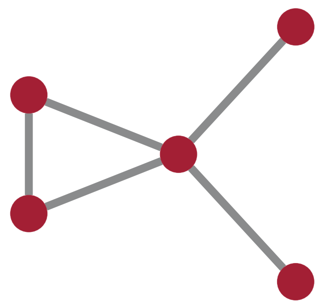


$$\mathcal{S} \simeq \sum_{G \in \mathcal{G}} s_G \mathrm{EFP}_G, \quad \mathcal{G} \text{ a set of multigraphs}$$

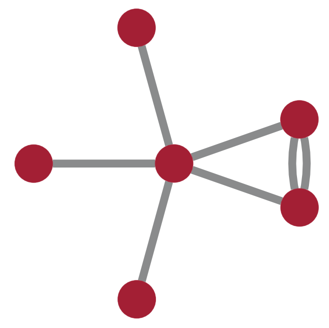


IRC-Safe Observable

Multivariate combinations of EFPs only require linear methods to achieve full generality



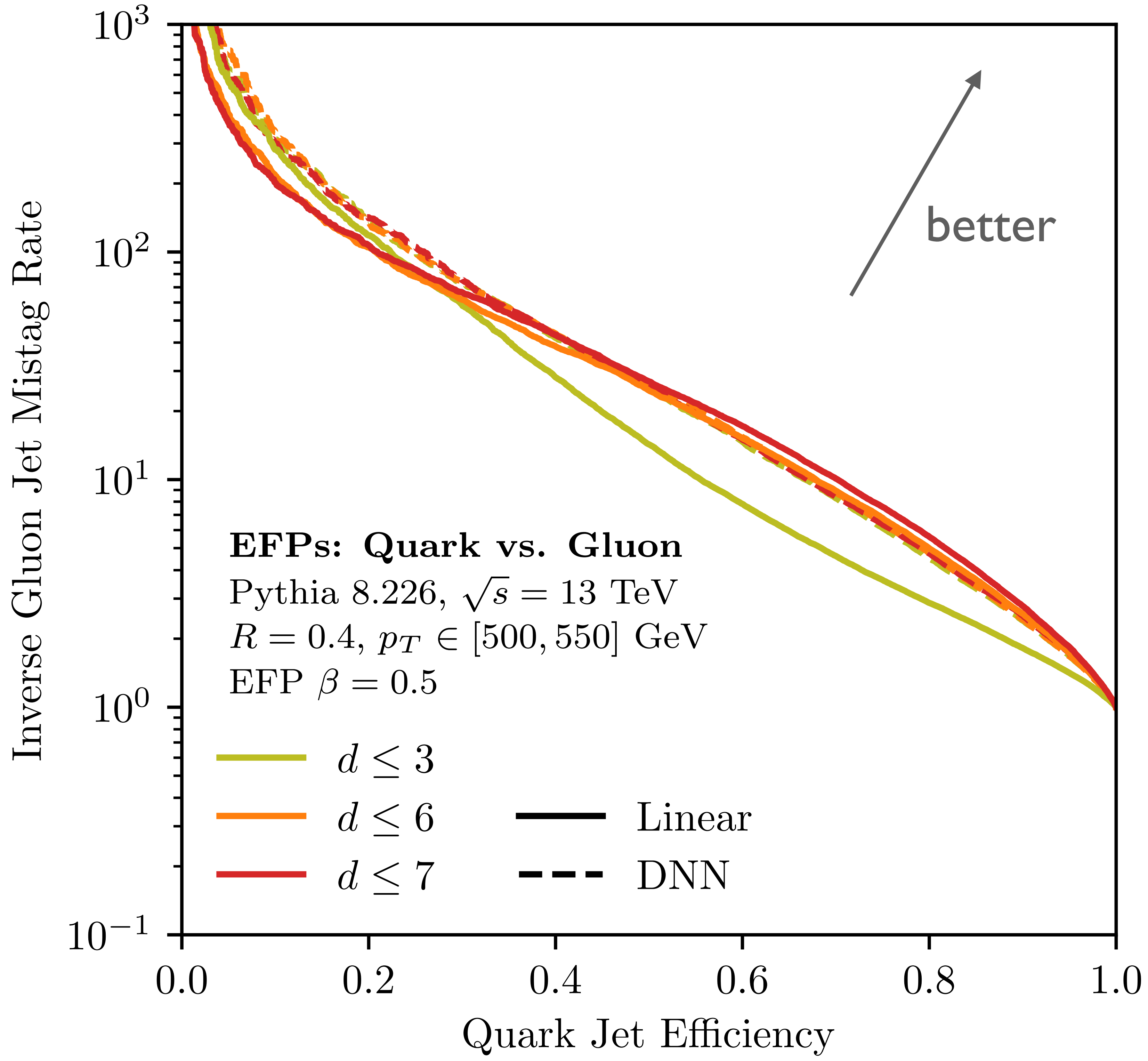
Strategy: Learn coefficients s_G via linear regression or classification



Testing EFPs on QCD Jets

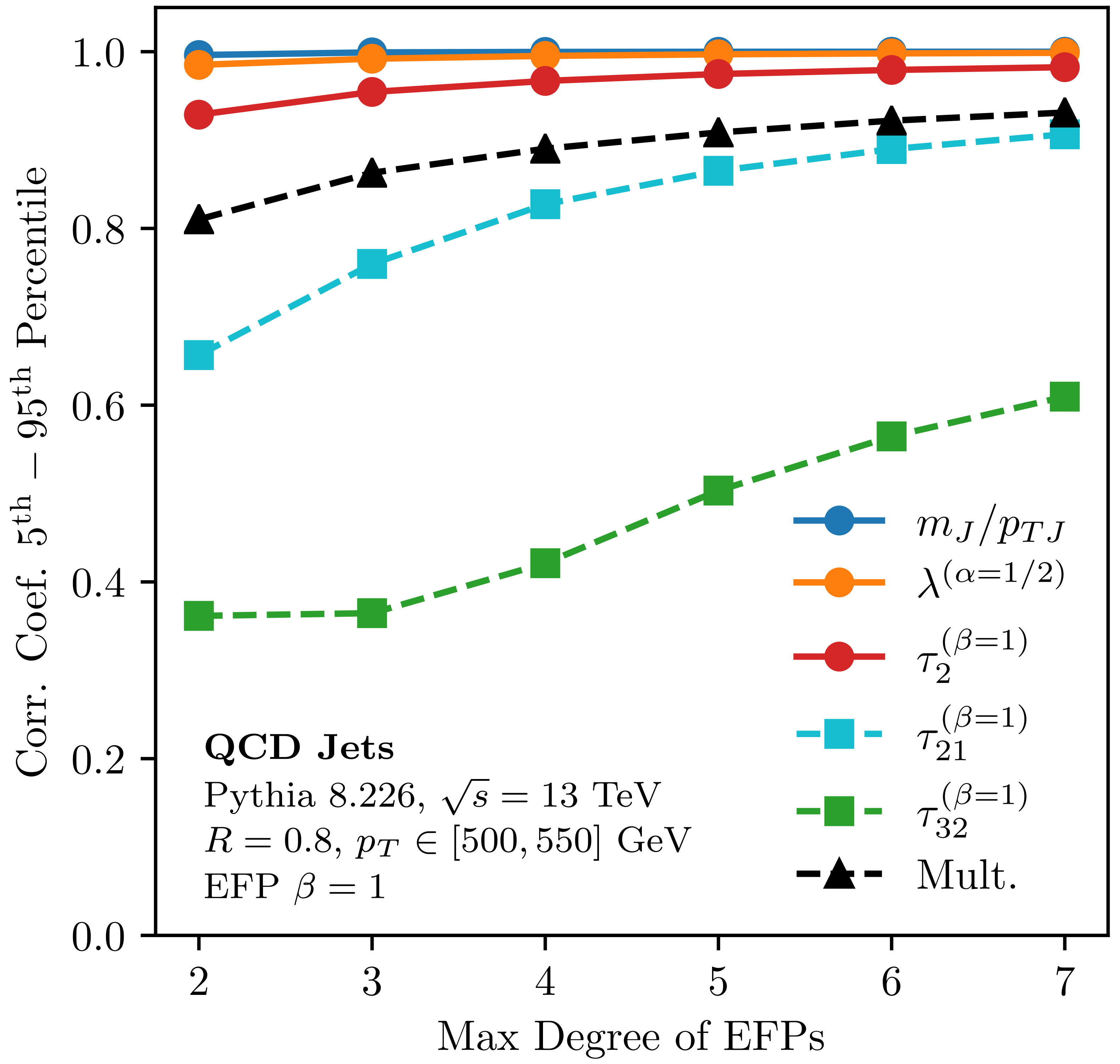
[PTK, Metodiev, Thaler, JHEP 2018]

Quark vs. Gluon Likelihood Ratio



Convergence observed with more EFPs

DNN gets there faster but linear suffices



EFPs regress onto IRC-safe observables best

Computational Complexity

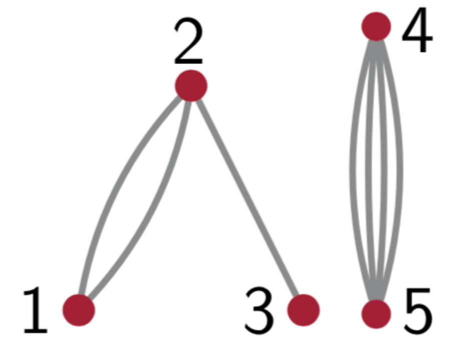
Naive computation complexity of an energy correlator is $\mathcal{O}(M^N)$

EnergyCorrelator fjcontrib solution:

```
// if N > 5, then throw error
if (_N > 5) {
    throw Error("EnergyCorrelator is only hard coded for N = 0,1,2,3,4,5");
}
```

Variable elimination (VE) algorithm: $\mathcal{O}(M^\chi)$, $\chi \lesssim N$

Disconnected is product of connected



$$= \left(\sum_{i_1=1}^{M}\sum_{i_2=1}^{M}\sum_{i_3=1}^{M} z_{i_1} z_{i_2} z_{i_3} \theta_{i_1 i_2}^2 \theta_{i_2 i_3}\right)\left(\sum_{i_4=1}^{M}\sum_{i_5=1}^{M} z_{i_4} z_{i_5} \theta_{i_4 i_5}^4\right)$$

VE find clever parentheses placement to minimize computation

$$= \underbrace{\sum_{i_1=1}^{M}\sum_{i_2=1}^{M}\sum_{i_3=1}^{M}\sum_{i_4=1}^{M}\sum_{i_5=1}^{M}\sum_{i_6=1}^{M}\sum_{i_7=1}^{M}\sum_{i_8=1}^{M} z_{i_1} z_{i_2} z_{i_3} z_{i_4} z_{i_5} z_{i_6} z_{i_7} z_{i_8} \prod_{j=2}^{7} \theta_{i_1 i_j}}_{\mathcal{O}(M^8)}$$

$$= \underbrace{\sum_{i_1=1}^{M} z_{i_1} \left(\sum_{i_2=1}^{M} z_{i_2} \theta_{i_1 i_2}\right)^7}_{\mathcal{O}(M^2)}$$

All tree graphs become $\mathcal{O}(M^2)$

$\chi = N$ iff G is complete graph, ECFs still slow

Energy Flow Moments (EFMs)

[PTK, Metodiev, Thaler, PRD 2020]

$\theta_{ij} = \sqrt{2 n_i^\mu n_{j\mu}}$ $\quad \beta = 2$ removes square root

Factors of n_i^μ can be organized in optimal way

EFM$_v$ is a little group tensor with v indices

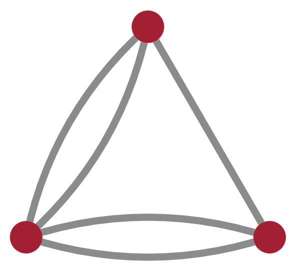
$$\mathcal{I}^{\mu_1 \cdots \mu_v} = \sum_{i=1}^{M} z_i n_i^{\mu_1} \cdots n_i^{\mu_v}$$

v	0	1	2	3	4	5	6
$n_{\text{components}}^{(d=4)}$	1	4	10	20	35	56	84

$$\mathcal{I}^{j_1 j_2 \cdots j_v} = 2^{v/2} \Theta^{j_1 j_2 \cdots j_v}$$

spatial e⁺e⁻ EFMs $\qquad$ linearized sphericity tensors

[Donoghue, Low, Pi, PRD 1979]



$$= \sum_{i_1=1}^{M}\sum_{i_2=1}^{M}\sum_{i_3=1}^{M} z_{i_1} z_{i_2} z_{i_3} \theta_{i_1 i_2}^2 \theta_{i_1 i_3}^2 \theta_{i_2 i_3}$$

$$= 2^5 \underbrace{\left(\sum_{i_1=1}^{M} z_{i_1} n_{i_1}^\alpha n_{i_1}^\beta n_{i_1}^\gamma n_{i_1}^\delta\right)}_{\mathcal{I}^{\alpha\beta\gamma\delta}} \underbrace{\left(\sum_{i_2=1}^{M} z_{i_2} n_{i_2\alpha} n_{i_2\beta} n_{i_2}^\epsilon\right)}_{\mathcal{I}_{\alpha\beta}{}^{\epsilon}} \underbrace{\left(\sum_{i_3=1}^{M} z_{i_3} n_{i_3\gamma} n_{i_3\delta} n_{i_3\epsilon}\right)}_{\mathcal{I}_{\gamma\delta\epsilon}}$$

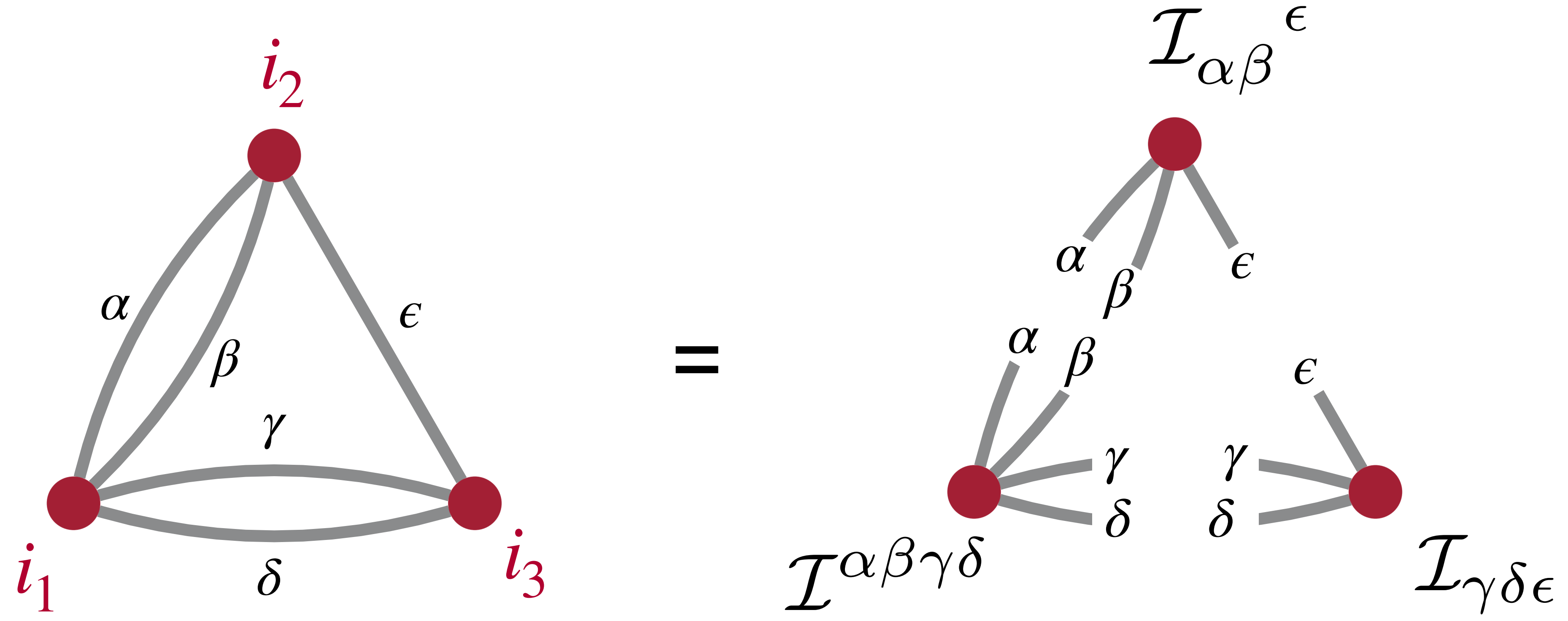
Naively $\mathcal{O}(M^3)$ EFP shown to be $\mathcal{O}(M)$

Computational Complexity

All $\beta = 2$ EFPs are $\mathcal{O}(M)$

- ECF$_N^{(\beta=2)}$ are all $\mathcal{O}(M)$
- $D_2^{(\beta=2)}, C_2^{(\beta=2)}$ are $\mathcal{O}(M)$

EFMs result from cutting edges of EFP graph



EFP

contractions of EFMs

Energy Flow Moments (EFMs)

[PTK, Metodiev, Thaler, PRD 2020]

$\theta_{ij} = \sqrt{2 n_i^\mu n_{j\mu}}$ $\beta = 2$ removes square root

Factors of n_i^μ can be organized in optimal way

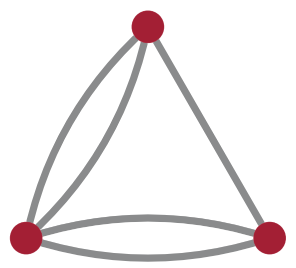
EFM$_v$ is a little group tensor with v indices

$$\mathcal{I}^{\mu_1 \cdots \mu_v} = \sum_{i=1}^{M} z_i n_i^{\mu_1} \cdots n_i^{\mu_v}$$

v	0	1	2	3	4	5	6
$n_{\text{components}}^{(d=4)}$	1	4	10	20	35	56	84

$$\mathcal{I}^{j_1 j_2 \cdots j_v} = 2^{v/2} \Theta^{j_1 j_2 \cdots j_v}$$

spatial e$^+$e$^-$ EFMs — linearized sphericity tensors [Donoghue, Low, Pi, PRD 1979]



$$= \sum_{i_1=1}^{M} \sum_{i_2=1}^{M} \sum_{i_3=1}^{M} z_{i_1} z_{i_2} z_{i_3} \theta_{i_1 i_2}^2 \theta_{i_1 i_3}^2 \theta_{i_2 i_3}$$

$$= 2^5 \underbrace{\left(\sum_{i_1=1}^{M} z_{i_1} n_{i_1}^{\alpha} n_{i_1}^{\beta} n_{i_1}^{\gamma} n_{i_1}^{\delta} \right)}_{\mathcal{I}^{\alpha\beta\gamma\delta}} \underbrace{\left(\sum_{i_2=1}^{M} z_{i_2} n_{i_2\alpha} n_{i_2\beta} n_{i_2}^{\epsilon} \right)}_{\mathcal{I}_{\alpha\beta}{}^{\epsilon}} \underbrace{\left(\sum_{i_3=1}^{M} z_{i_3} n_{i_3\gamma} n_{i_3\delta} n_{i_3\epsilon} \right)}_{\mathcal{I}_{\gamma\delta\epsilon}}$$

Naively $\mathcal{O}(M^3)$ EFP shown to be $\mathcal{O}(M)$

Understanding Linear Redundancies via EFMs

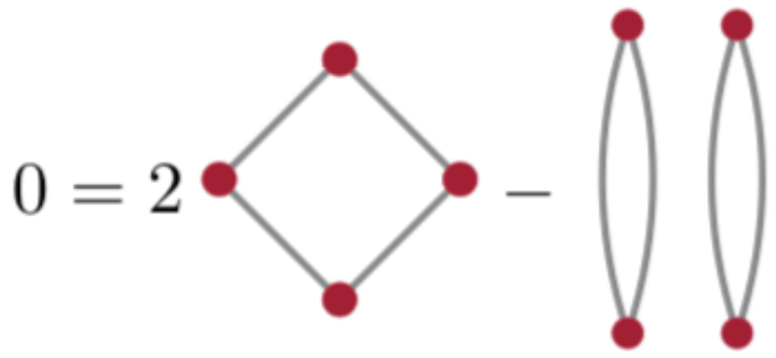
[PTK, Metodiev, Thaler, PRD 2020]

Linear redundancies among EFPs are troublesome

Studying coefficients of linear fit difficult $\mathcal{O} = \sum_G s_G \mathrm{EFP}_G$

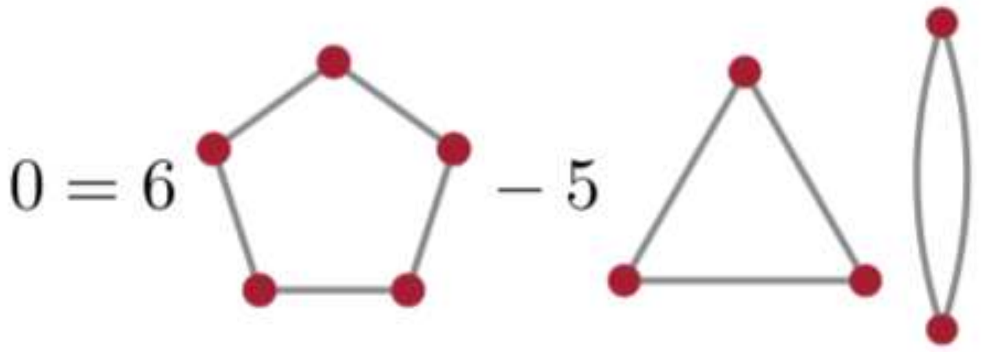
Examples of redundancies

in 3 or fewer spacetime dimensions

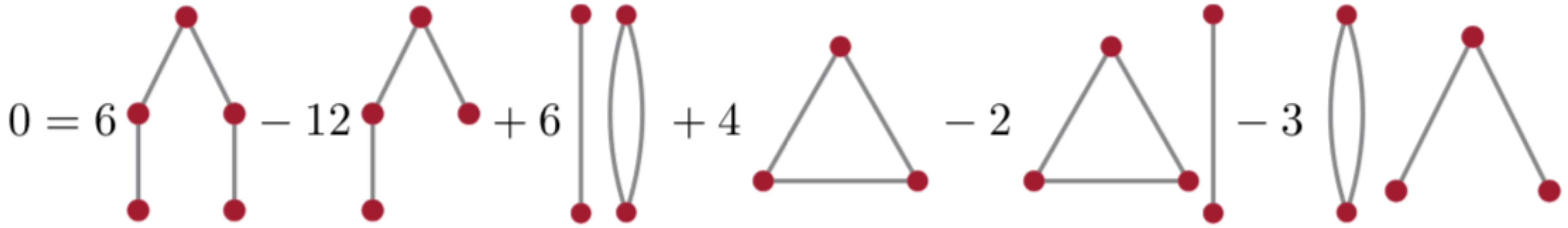


$$0 = \mathcal{I}^{\beta}_{[\alpha}\mathcal{I}^{\gamma}_{\beta}\mathcal{I}^{\delta}_{\gamma}\mathcal{I}^{\alpha}_{\delta]}$$

in 4 or fewer spacetime dimensions



$$0 = \mathcal{I}^{\beta}_{[\alpha}\mathcal{I}^{\gamma}_{\beta}\mathcal{I}^{\delta}_{\gamma}\mathcal{I}^{\epsilon}_{\delta}\mathcal{I}^{\alpha}_{\epsilon]}$$



How to obtain a tensor identity

Consider tensor over *n* dimensional vector space

Antisymmetrize *m* > *n* indices

Result is zero because any assignment of *n* possible values to *m* slots has a repetition

$$T^{a_1 \cdots a_k}_{b_1 \cdots b_\ell [c_1 \cdots c_m]} = 0$$

Bonus: all tensor identities up to ones governed by existing symmetries take above form

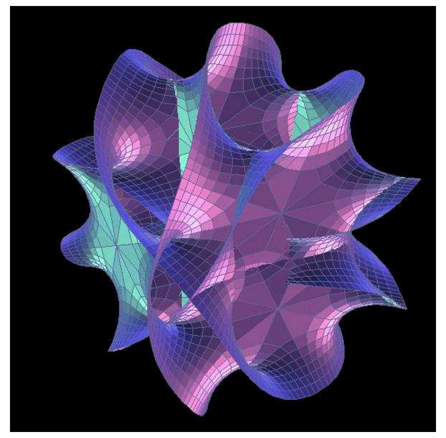
[Sneddon, Journal of Mathematical Physics]

In e^+e^- there are additional relations due to

$$n_i^\mu = (1, \hat{n})^\mu \quad \Longrightarrow \quad \mathcal{I}^{0\mu_1 \cdots \mu_v} = \sqrt{2}\mathcal{I}^{\mu_1 \cdots \mu_v}$$

See backup for more on these "Euclidean" relations

Counting Superstring Amplitudes



New OEIS Entries!
A307317, A307316

Constructing a basis of amplitudes – how large is it?

[Boels, 1304.7918; OEIS A226919]

non-isomorphic multigraph

$\Updownarrow$

Q: What is the number of symmetric polynomials of degree d in kinematic variables up to momentum conservation?

$\Updownarrow$ $\Updownarrow$

$$\theta_{ij}^2 = 2n_i \cdot n_j$$

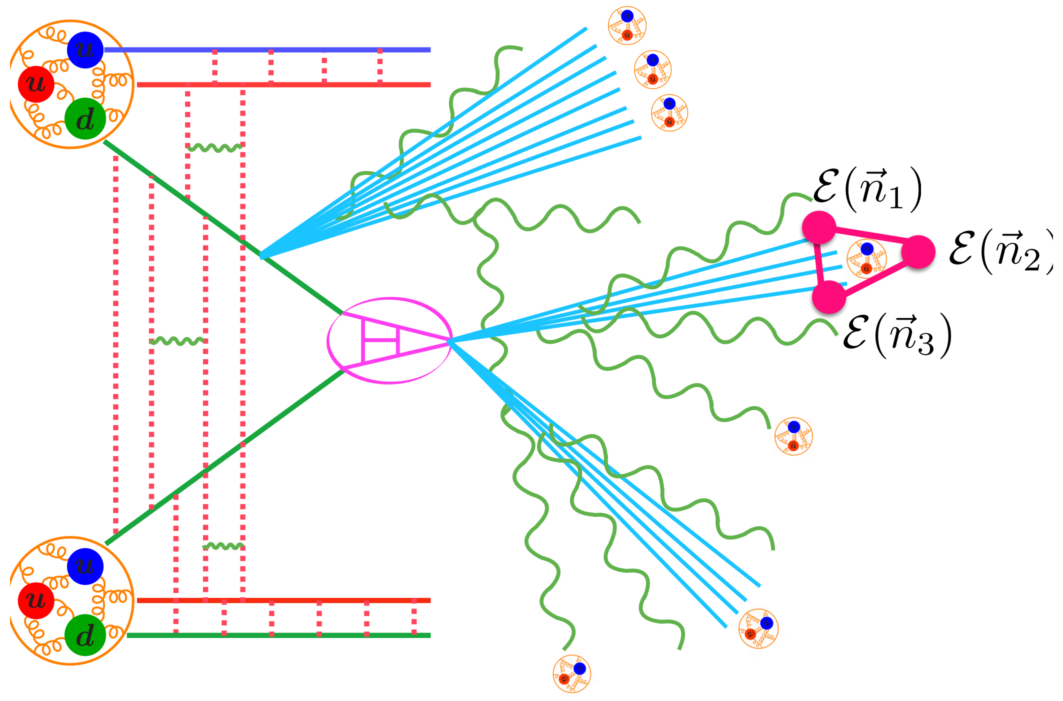
pairwise angular distance

$$0 = \sum_{i=1}^{M} p_i^\mu = \mathcal{I}^\mu =$$ 

A: Same as the number of non-isomorphic multigraphs with no leaves (vertices of valency one)

	Leafless Multigraphs	
	Connected	All
Edges d	A307317	A307316
1	0	0
2	1	1
3	2	2
4	4	5
5	9	11
6	26	34
7	68	87
8	217	279
9	**718**	**897**
10	**2 553**	**3 129**
11	**9 574**	**11 458**
12	**38 005**	**44 576**
13	**157 306**	**181 071**
14	**679 682**	**770 237**
15	**3 047 699**	**3 407 332**
16	**14 150 278**	**15 641 159**

Bolded values previously unknown

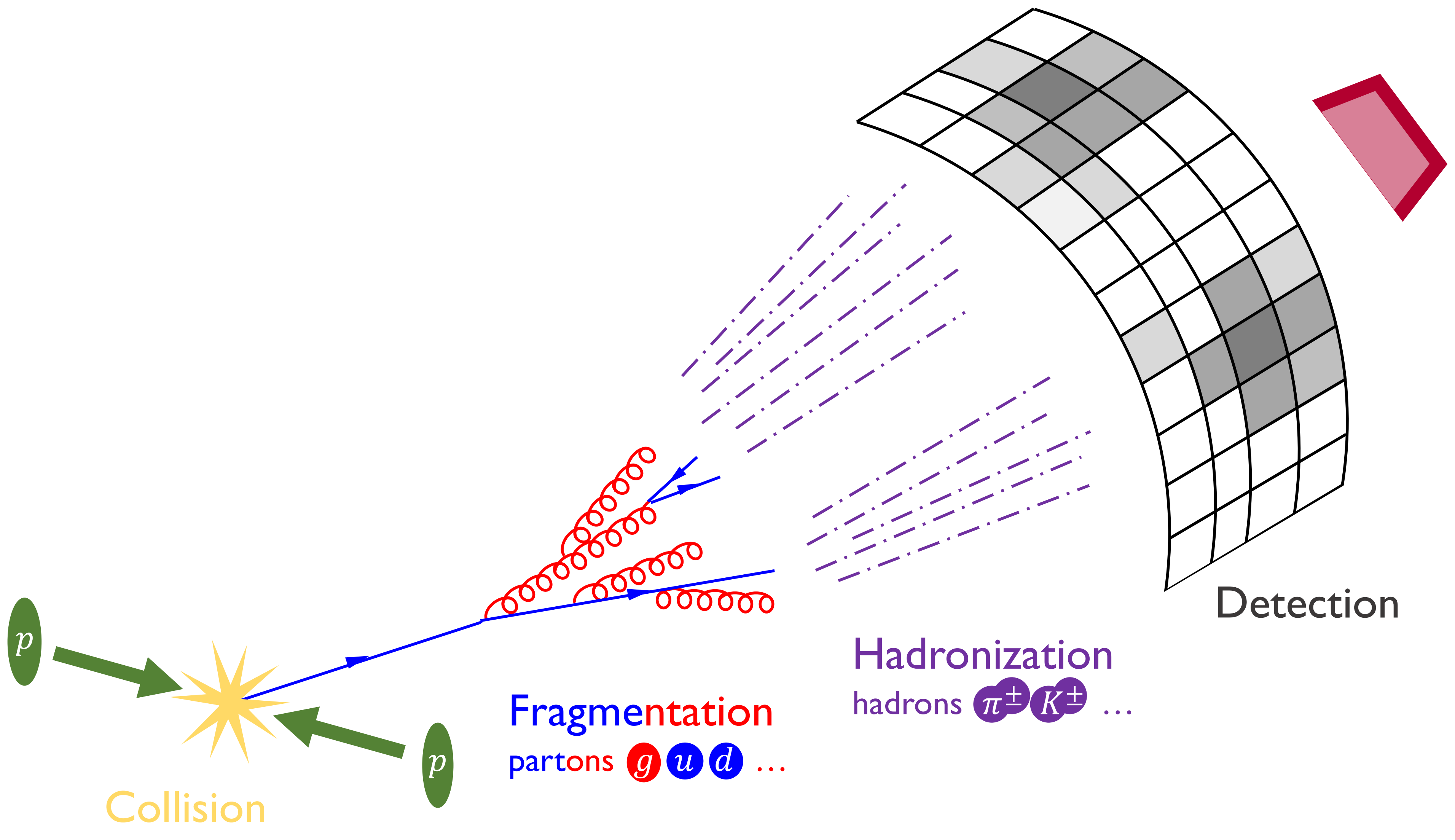


Collider Event Fundamentals

Energy Flow Observables

Energy-Energy Correlators

Beyond Observables via Weighted Cross Sections



$$\mathcal{E}(\hat{n}) = \int_0^\infty dt \lim_{r\to\infty} r^2 n^i T_{0i}(t, r\hat{n})$$

Stress-energy flow

Standard observable (e.g. EFPs)

Calculate a single number for each jet/event and study distribution of values

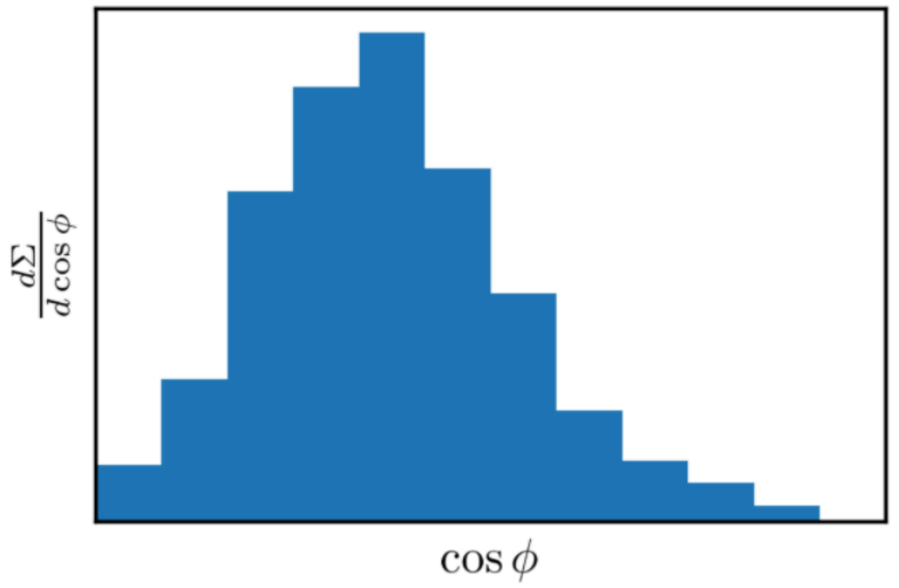
Weighted cross section

Calculate a distributional quantity per event and study the mean distribution

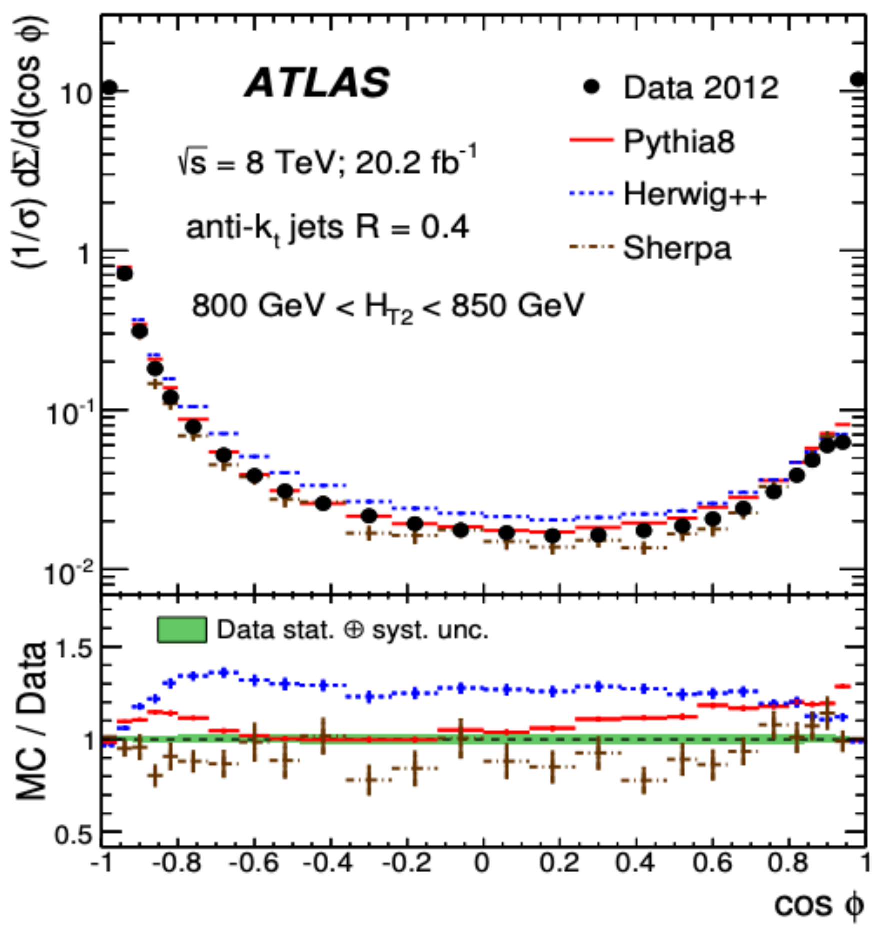
e.g. energy-energy correlator (EEC)

$$\frac{d\Sigma}{d\cos\phi} = \sum_{i,j} \int d\sigma \frac{E_i E_j}{Q^2} \delta(\cos\theta_{ij} - \cos\phi)$$

$$\Downarrow$$



$\Rightarrow$



$$\frac{1}{\sigma_{\text{tot}}} \frac{d\sigma}{d\hat{n}_1 \cdots d\hat{n}_N} = \frac{\langle \mathcal{O}\mathcal{E}(\hat{n}_1)\cdots\mathcal{E}(\hat{n}_N)\mathcal{O}^\dagger\rangle}{\langle \mathcal{O}\mathcal{O}^\dagger\rangle}$$

Correlations of energy flow operators can be directly studied!

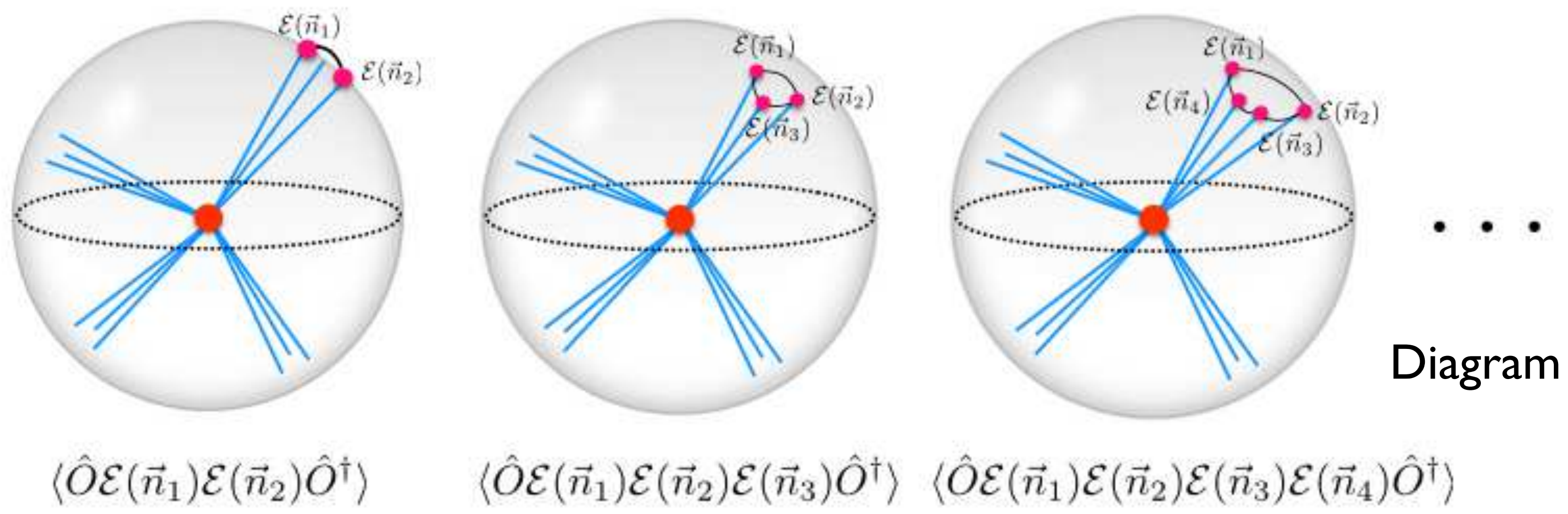


Diagram by Ian Moult

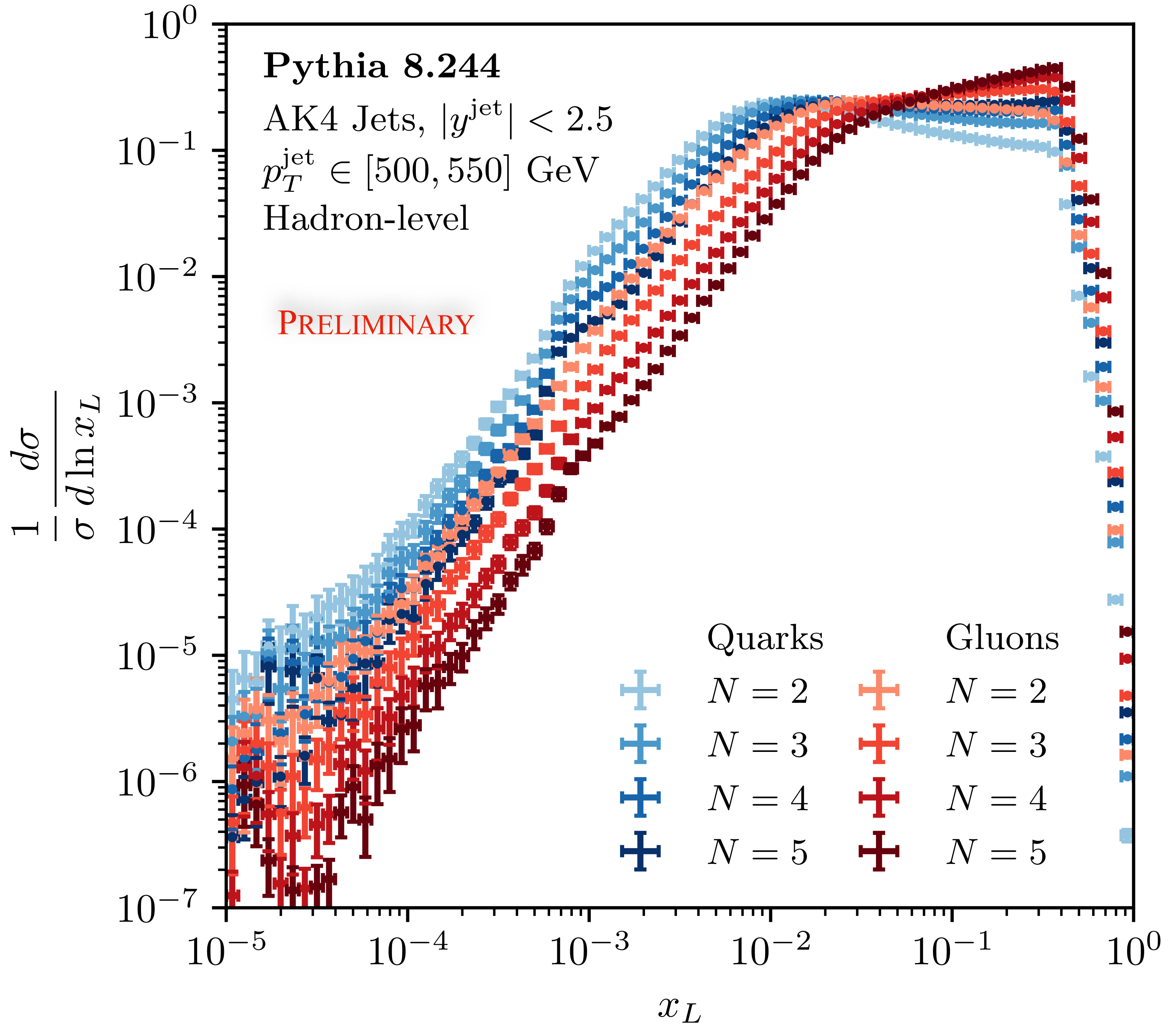
[PTK, Moult, Thaler, Zhu, *to appear soon*]

Energy-Energy Correlators – Projection to Longest Side

[PTK, Moult, Thaler, Zhu, *to appear soon*]

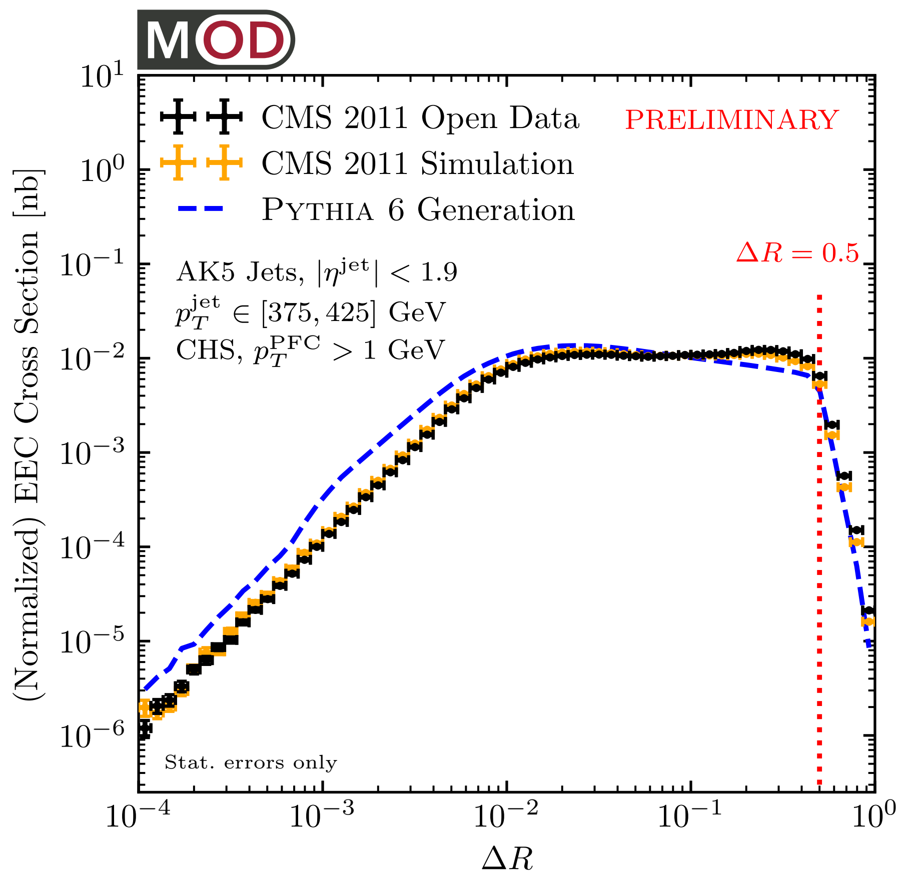
Integrate out shape dependence but keep overall size dependence

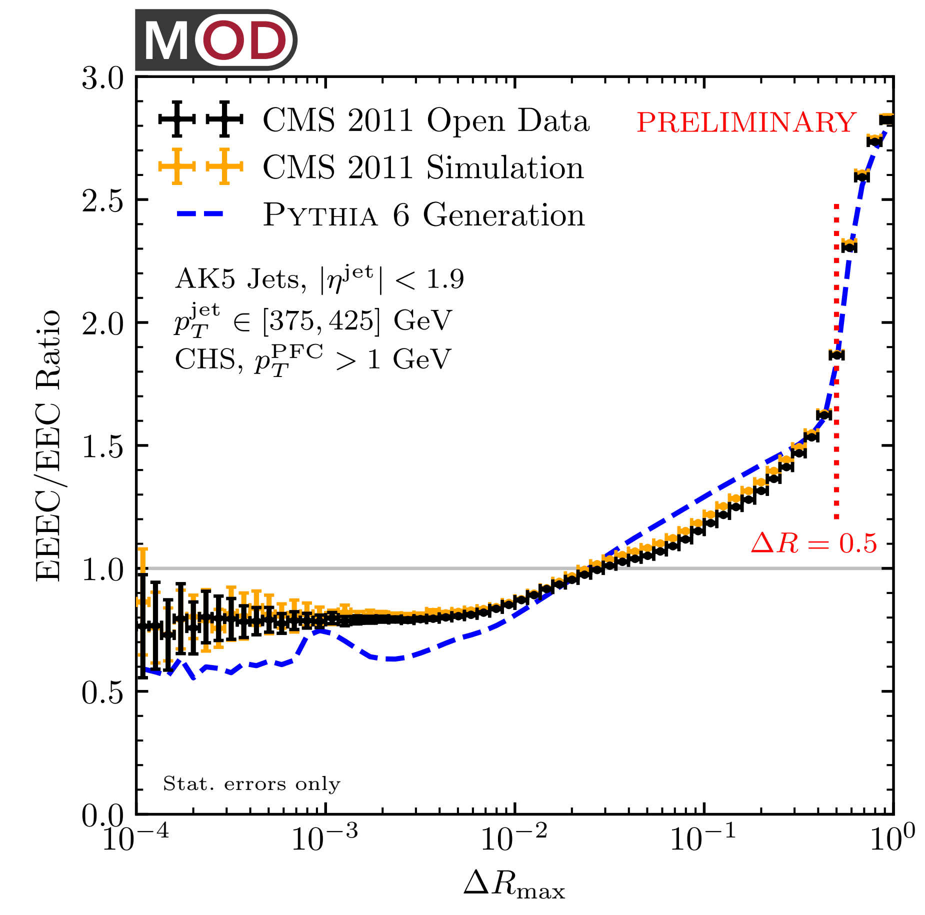
$$\frac{d\Sigma^{[N]}}{dx_L} = \sum_n \sum_{1\le i_1 \le \ldots \le i_N \le n} \int d\sigma_n \frac{E_{i_1}\cdots E_{i_N}}{Q^N}\delta(x_L - \max_{1\le j<k\le N}\{\theta_{i_j i_k}\})$$



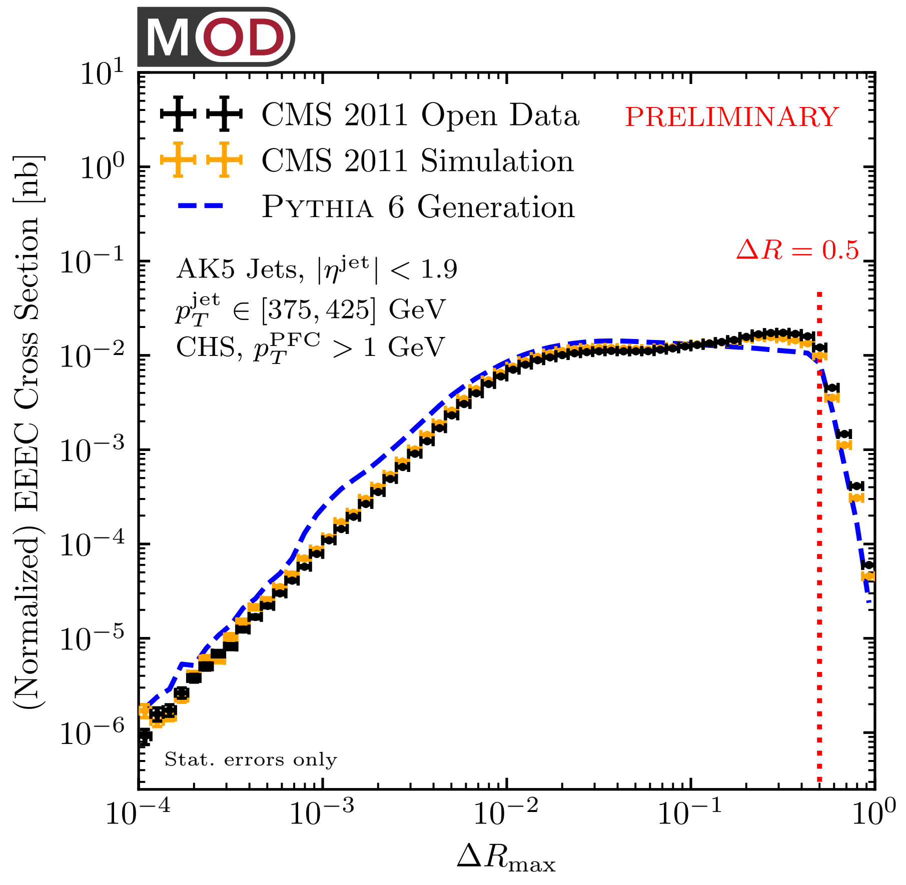
EEEC/EEC Ratio

$N = 2$

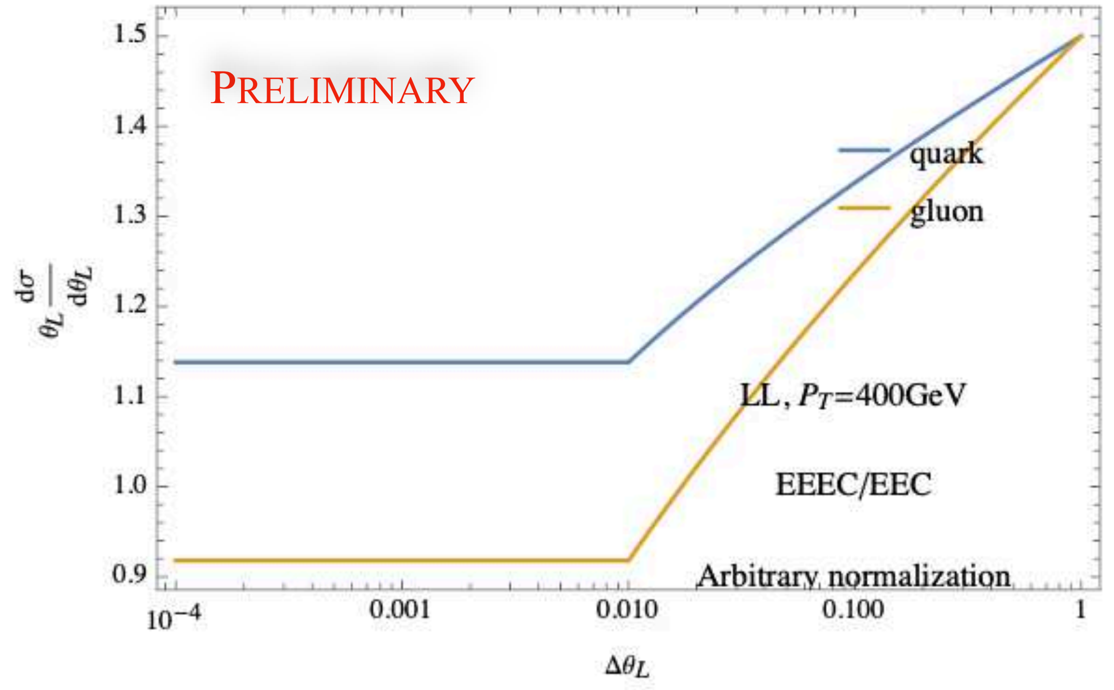




$N = 3$



LL prediction of ratio



EEEC – Full Shape Dependence

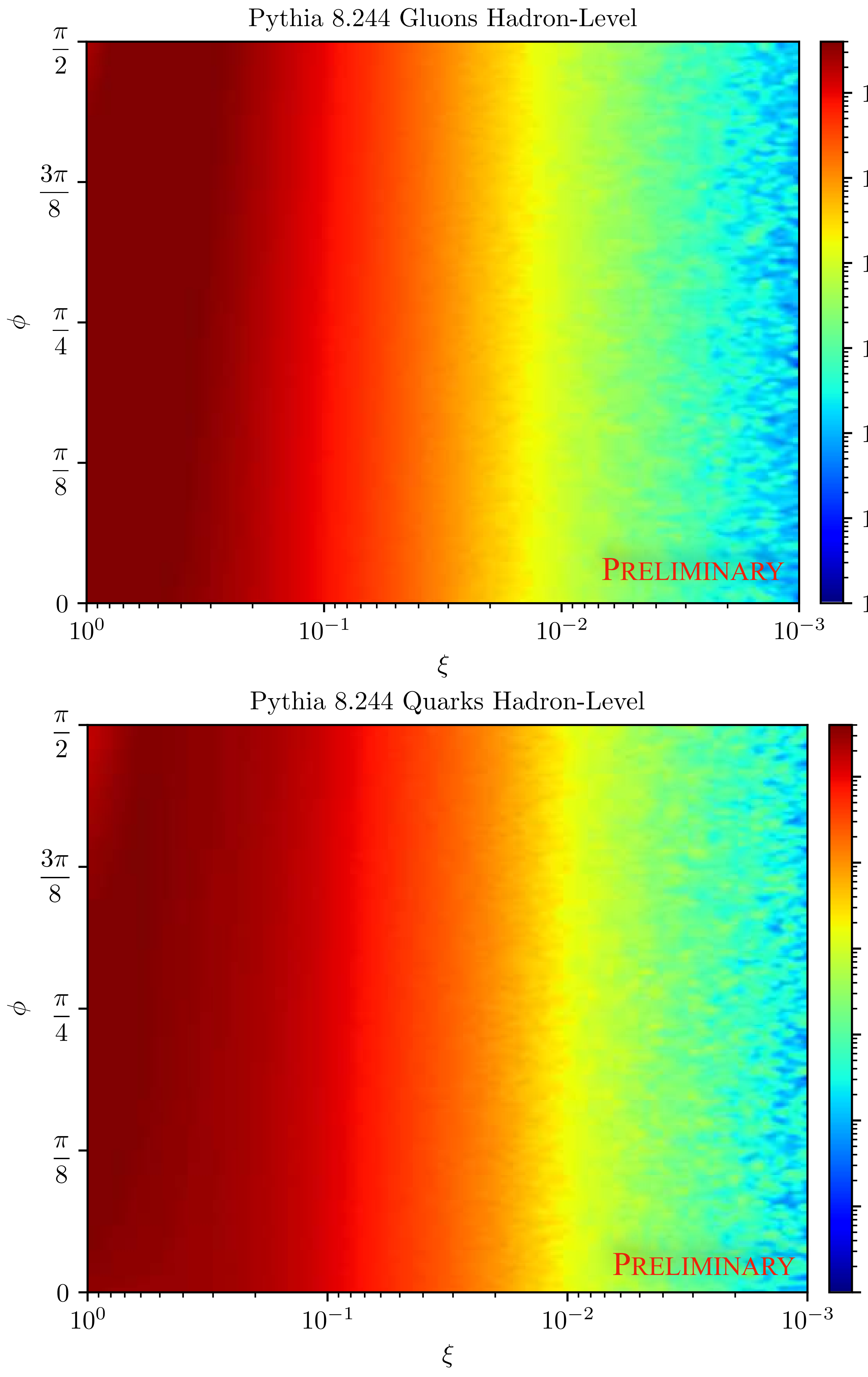
[PTK, Moult, Thaler, Zhu, *to appear soon*]

For $x_L \sim 0.01$

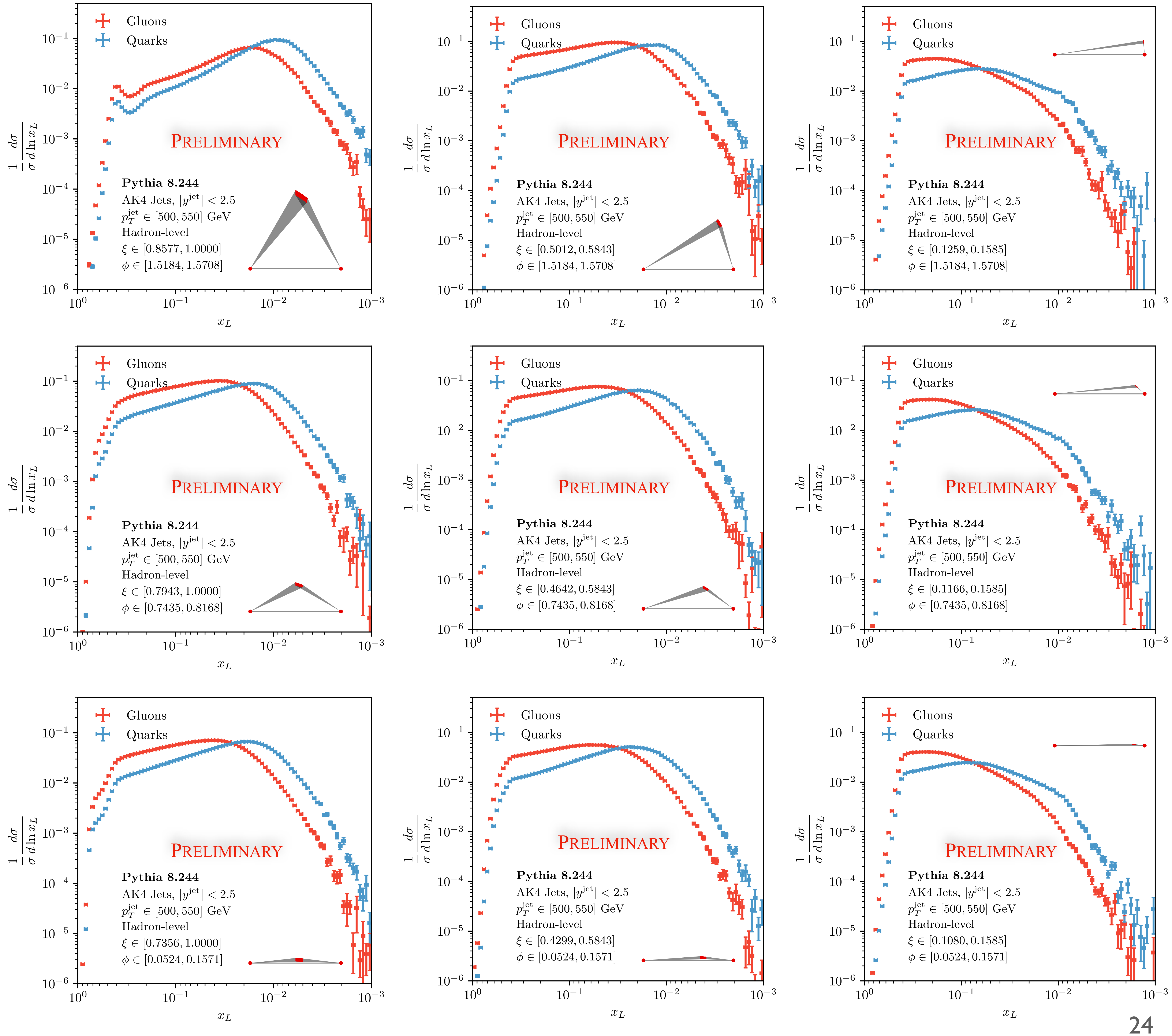
less aligned

ϕ

more aligned



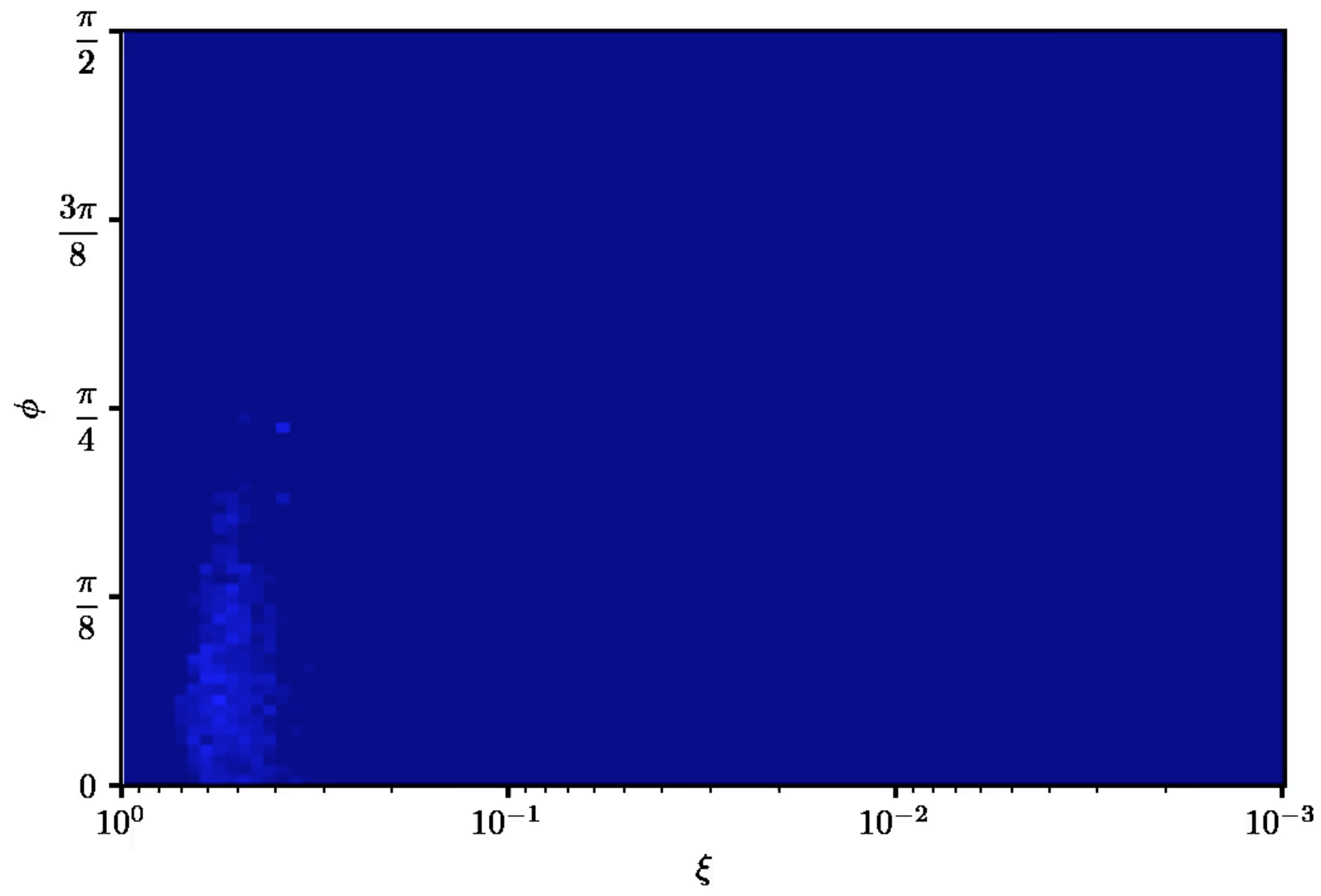
less collinear ⟷ more collinear



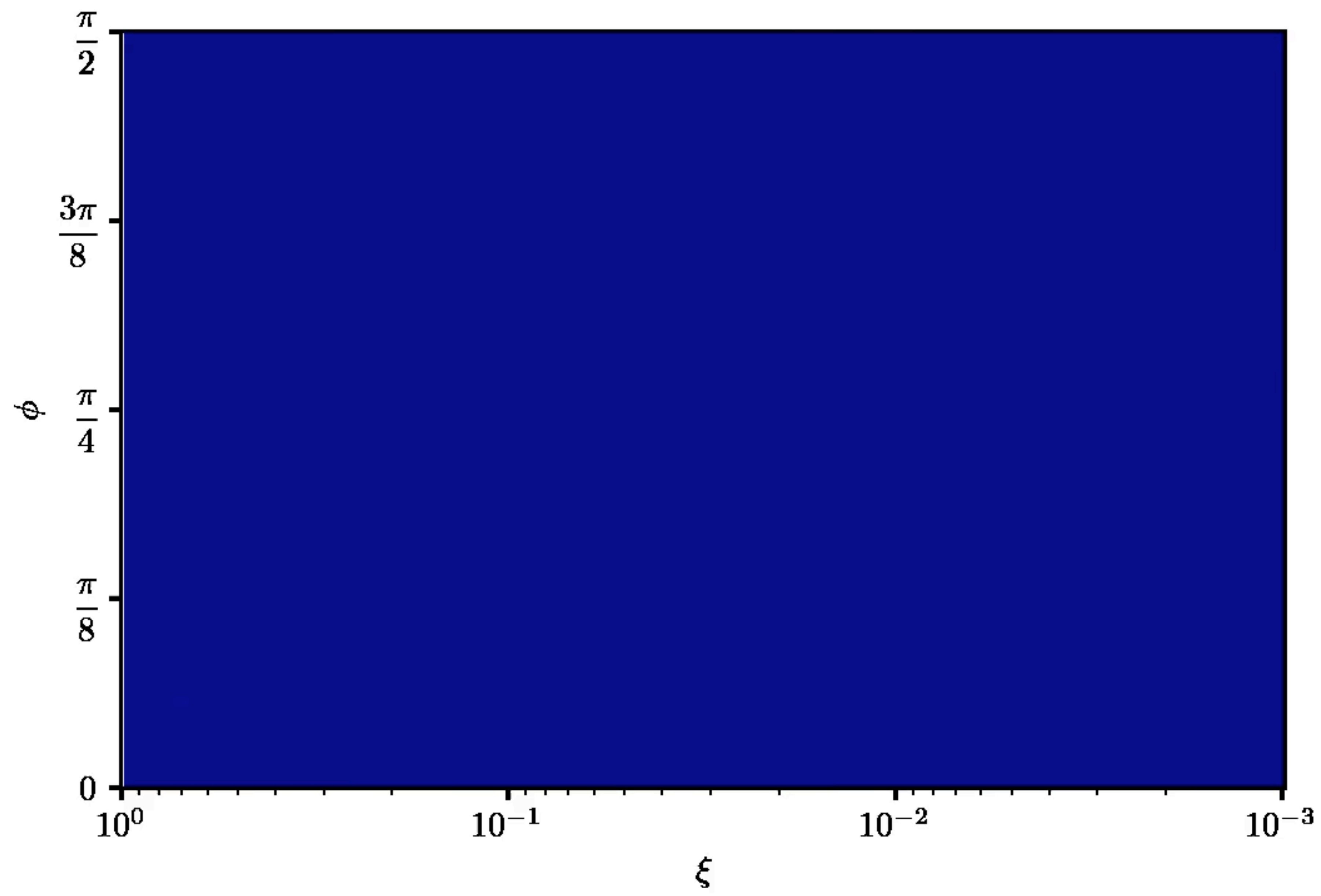
Visualizing the 3D EEEC

[PTK, Moult, Thaler, Zhu, *to appear soon*]

Pythia Gluon Jets
$p_T \in [500{,}550]$ GeV



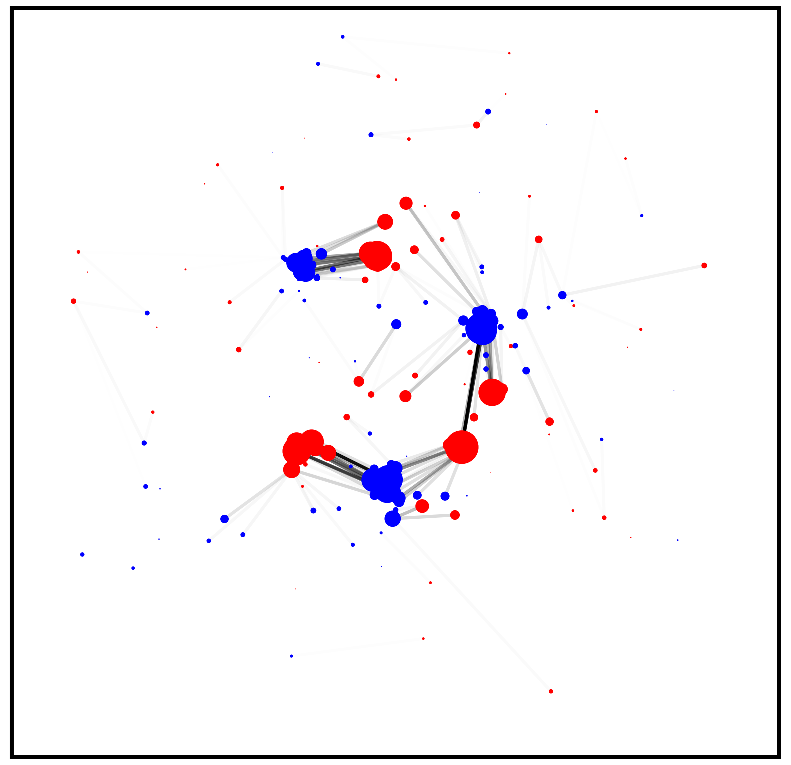
Pythia Quark Jets
$p_T \in [500{,}550]$ GeV

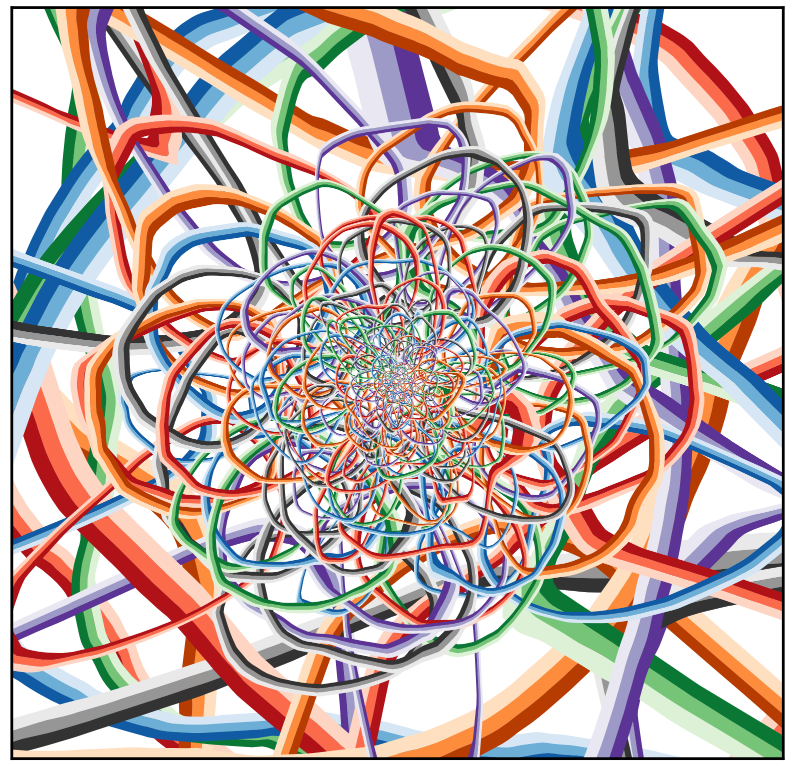


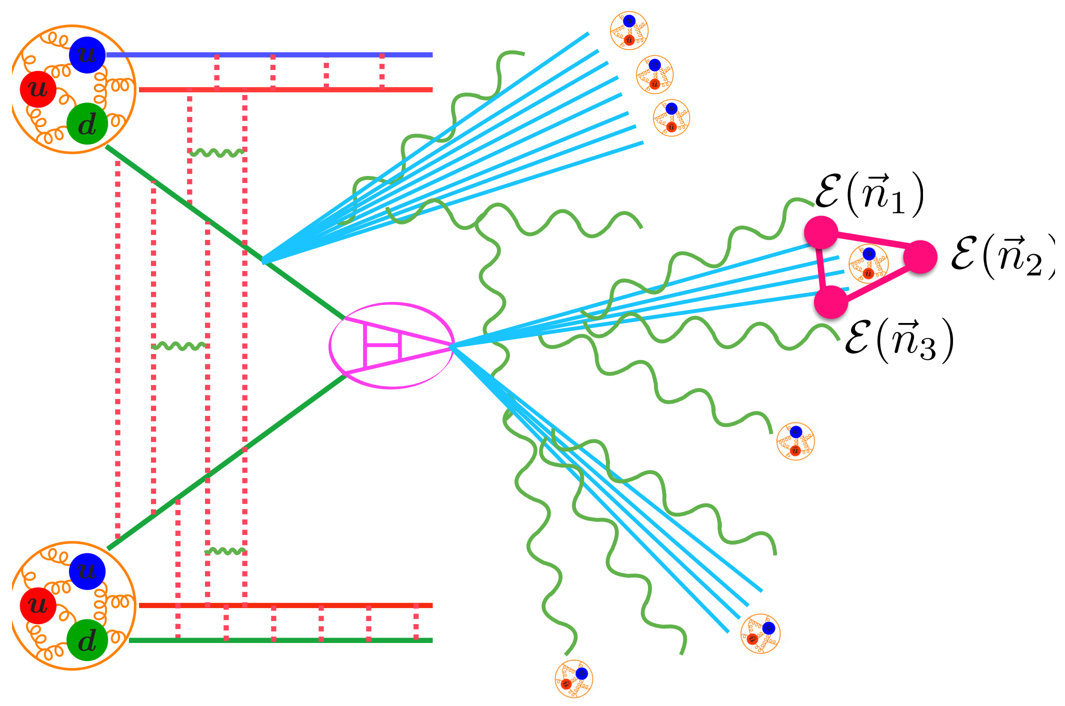
Time in the videos corresponds to $\ln x_L$ going from 0 to $-\infty$

Color corresponds to log of EEC (red is large, blue is small)

Uniformly persistent red is roughly the perturbatively accessible region







Collider Event Fundamentals

– Energy flow is theoretically and experimentally robust

– See 2004.04159 or this talk for a new geometric phrasing of many collider techniques

Energy Flow Observables

– EFPs provide a systematic basis of IRC observables and encompass

– EFMs allow for understanding EFP linear relations and fast computation

– EFNs (1810.05165) use ML to learn optimal IRC-safe observables automatically

Energy-Energy Correlators

Stay tuned!

– Single-log observables (no grooming!) probe detailed properties of QCD

– Benefit from jet substructure and conformal field theory insights

– Relatively simple to interface with track functions to compute on charged particles

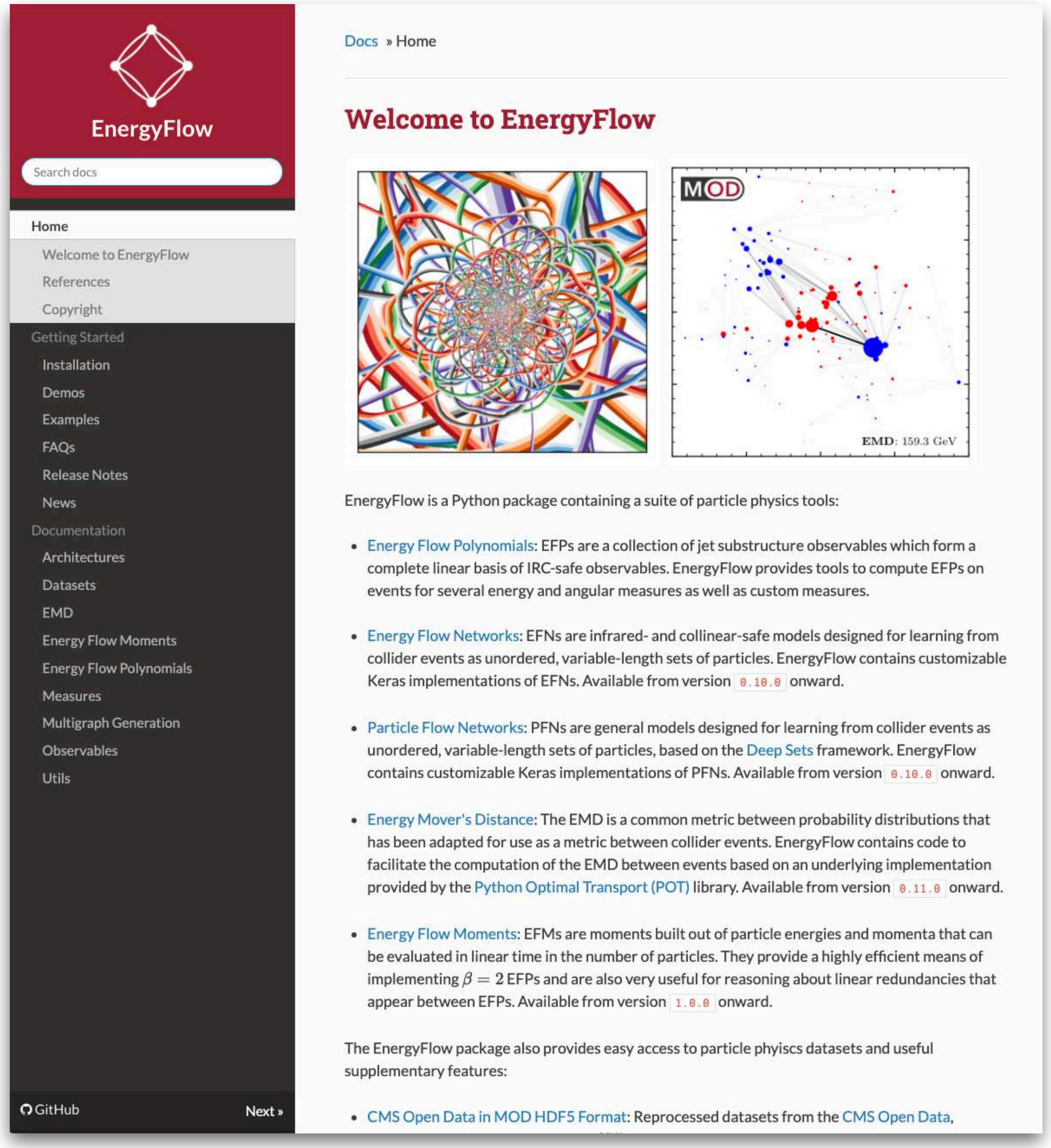
EnergyFlow Python Package

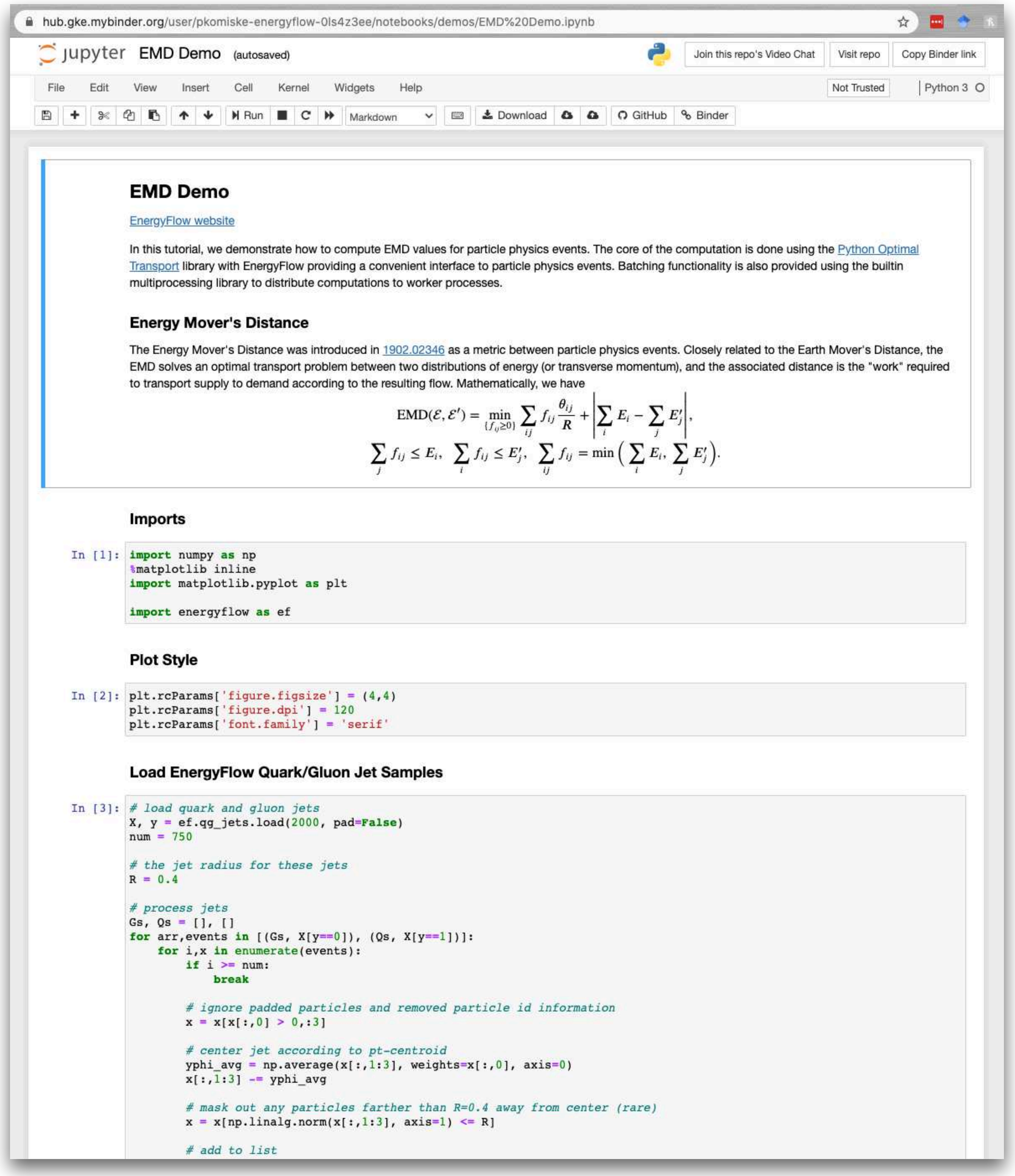
pip3 install energyflow

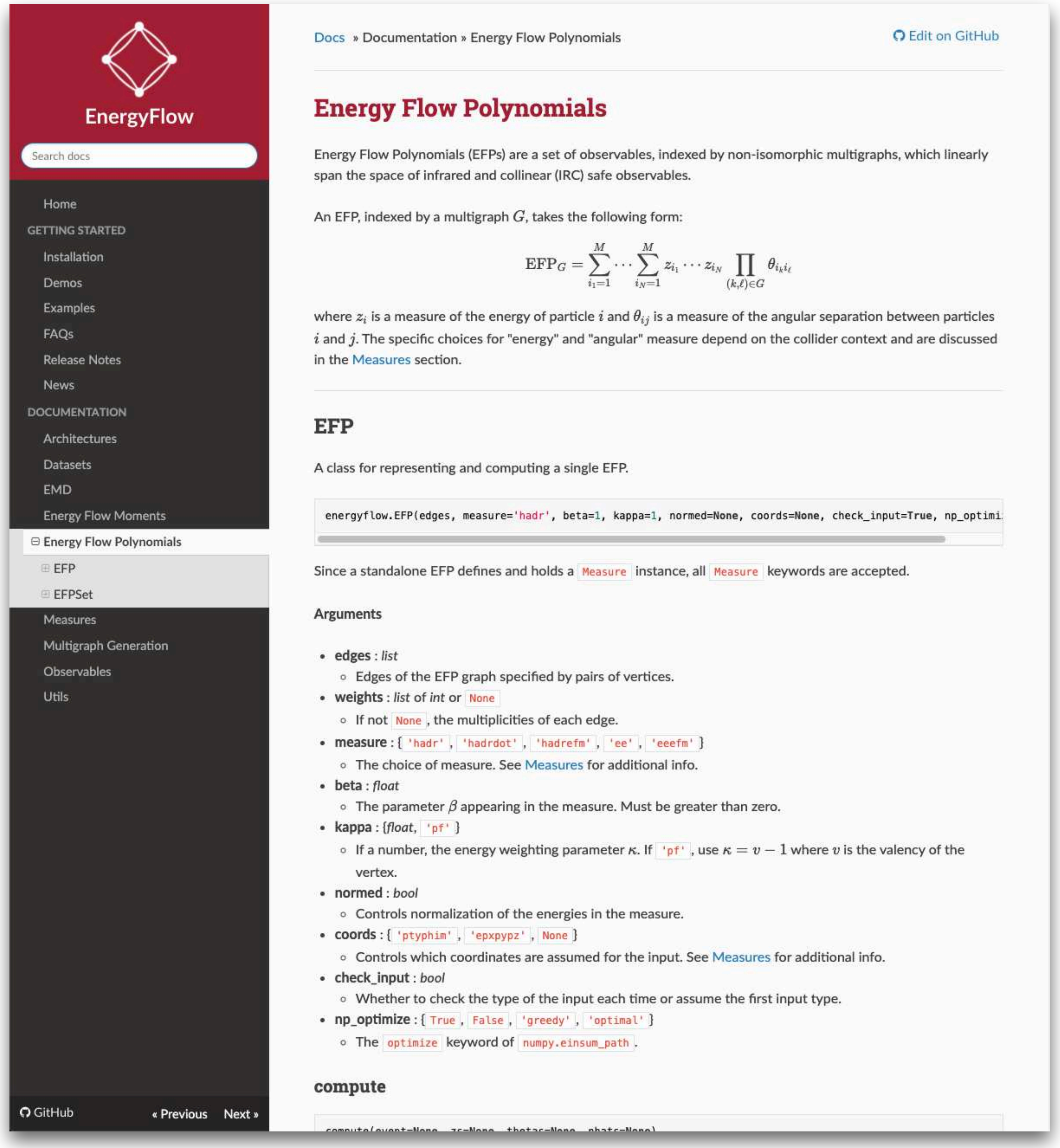
Computes EFPs efficiently via variable elimination and EFMs (for $\beta = 2$)

Detailed examples, demos, and documentation

Also computes EMD, constructs EFNs/PFNs, and loads datasets





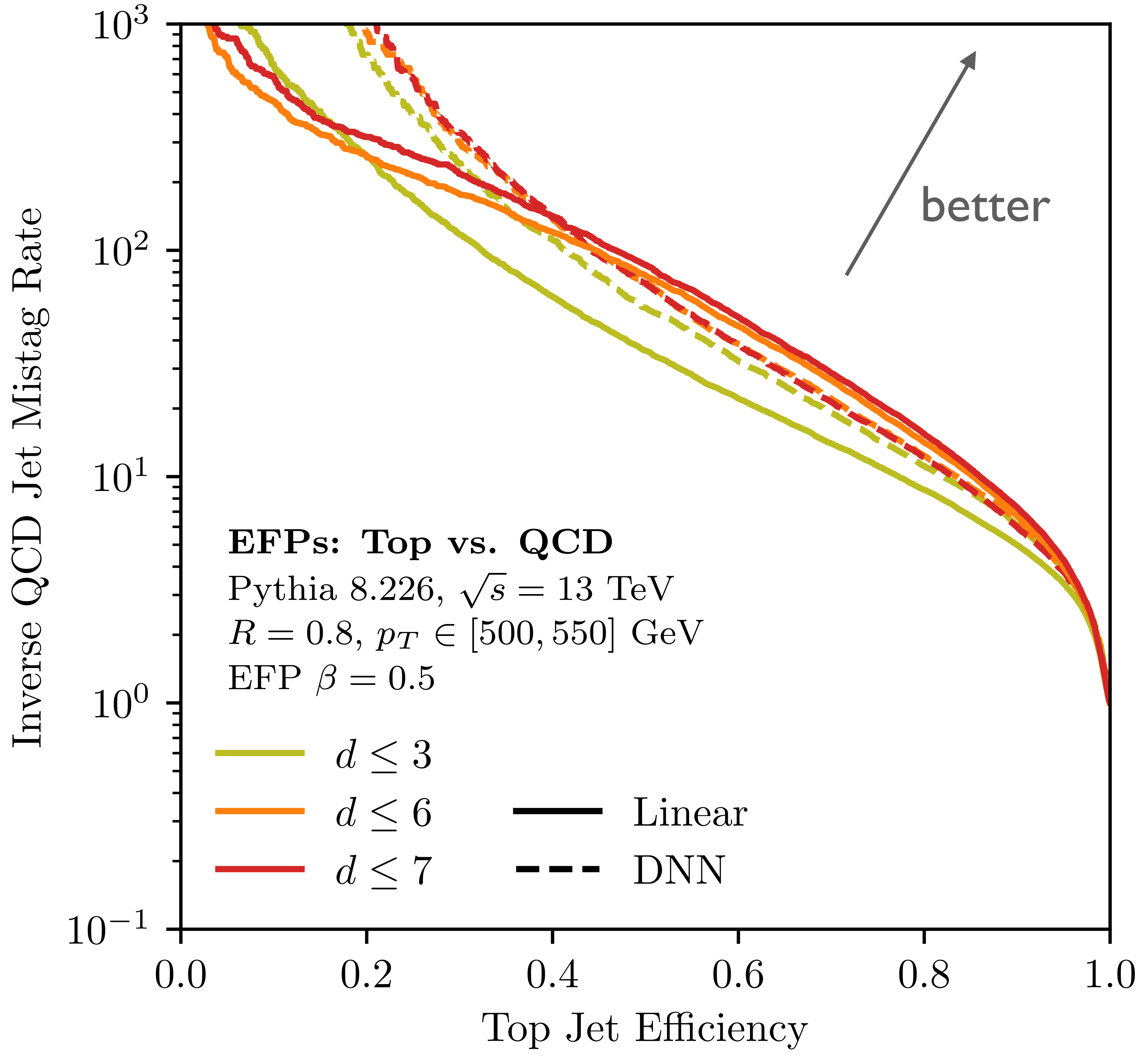


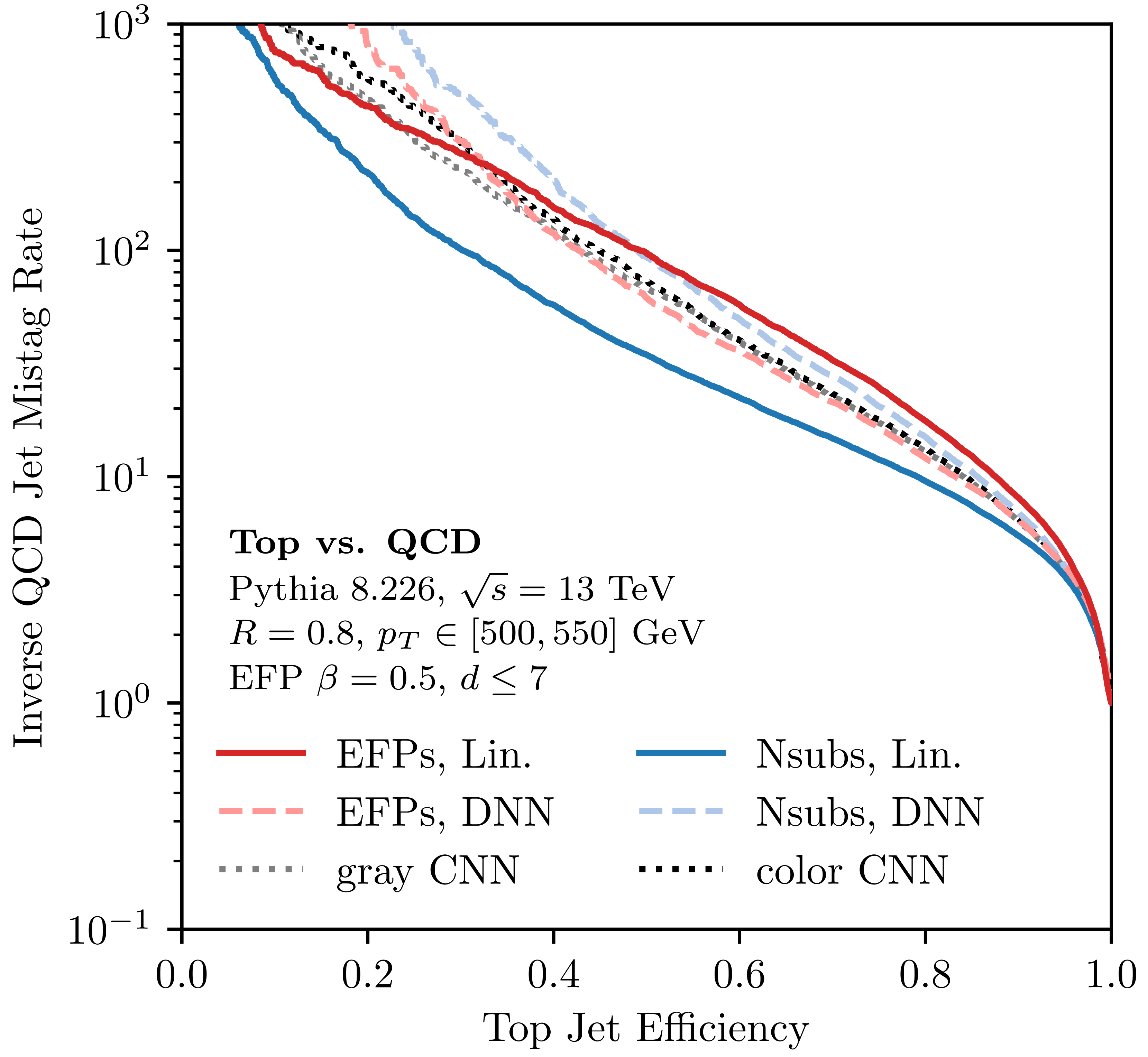
Additional Slides

Rewriting General EFP as Contraction of EFMs

$$\begin{aligned}
\mathrm{EFP}_G &= \sum_{i_1=1}^{M} \cdots \sum_{i_N=1}^{M} E_{i_1} \cdots E_{i_N} \prod_{(k,\ell)\in G} 2\eta_{\mu\nu} n_{i_k}^{\mu} n_{i_\ell}^{\nu} \\
&= \left(\prod_{j=1}^{N} \sum_{i_j=1}^{M} z_{i_j} n_{i_j}^{\mu_1^j} n_{i_j}^{\mu_2^j} \cdots n_{i_j}^{\mu_{v_j}^j} \right) \prod_{(k,\ell)\in G} 2\eta_{\mu^k_{A_{k\ell}} \mu^\ell_{A_{\ell k}}} \\
&= \left(\prod_{j=1}^{N} \mathcal{I}^{\mu_1^j \mu_2^j \cdots \mu_{v_j}^j} \right) \prod_{(k,\ell)\in G} 2\eta_{\mu^k_{A_{k\ell}} \mu^\ell_{A_{\ell k}}},
\end{aligned}$$

Boosted Top: EFP Classification Performance Comparison





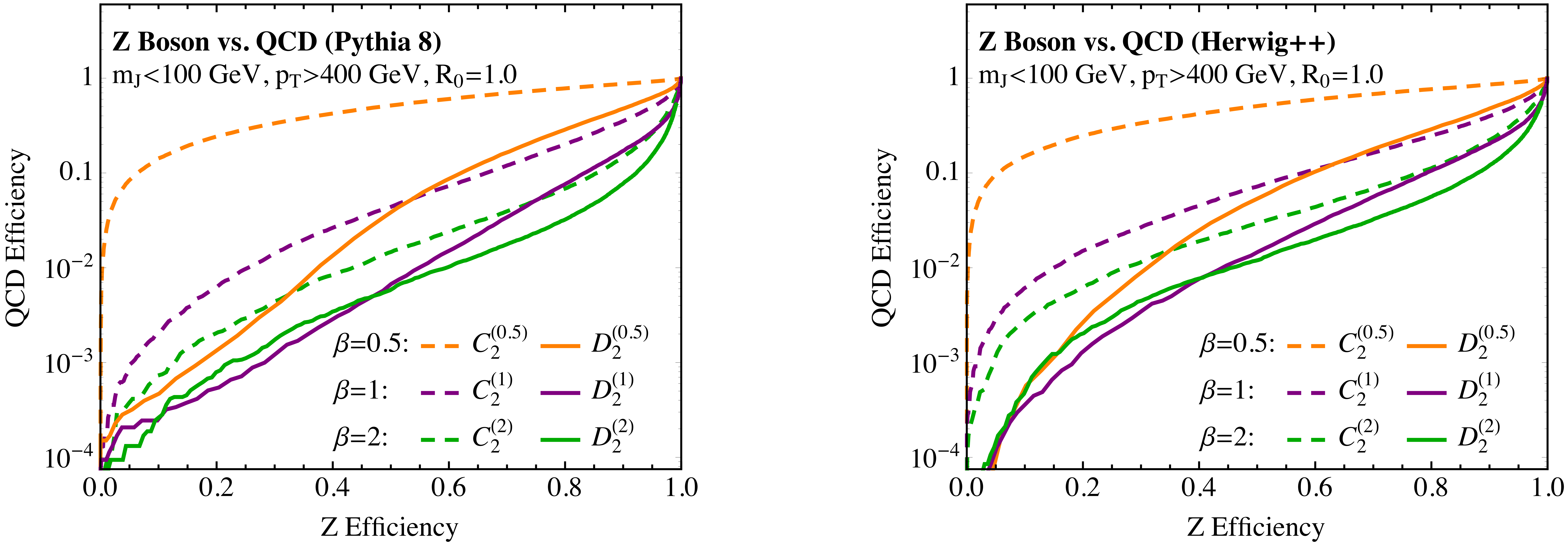
Saturation observed with more EFPs

Linear EFPs excel at high efficiency

DNN gets there faster but linear suffices

[de Oliviera, Kagan, Mackey, Nachman, Schwartzman, 2015; PTK, Metodiev, Schwartz, 2016; Datta, Larkoski, 2017]

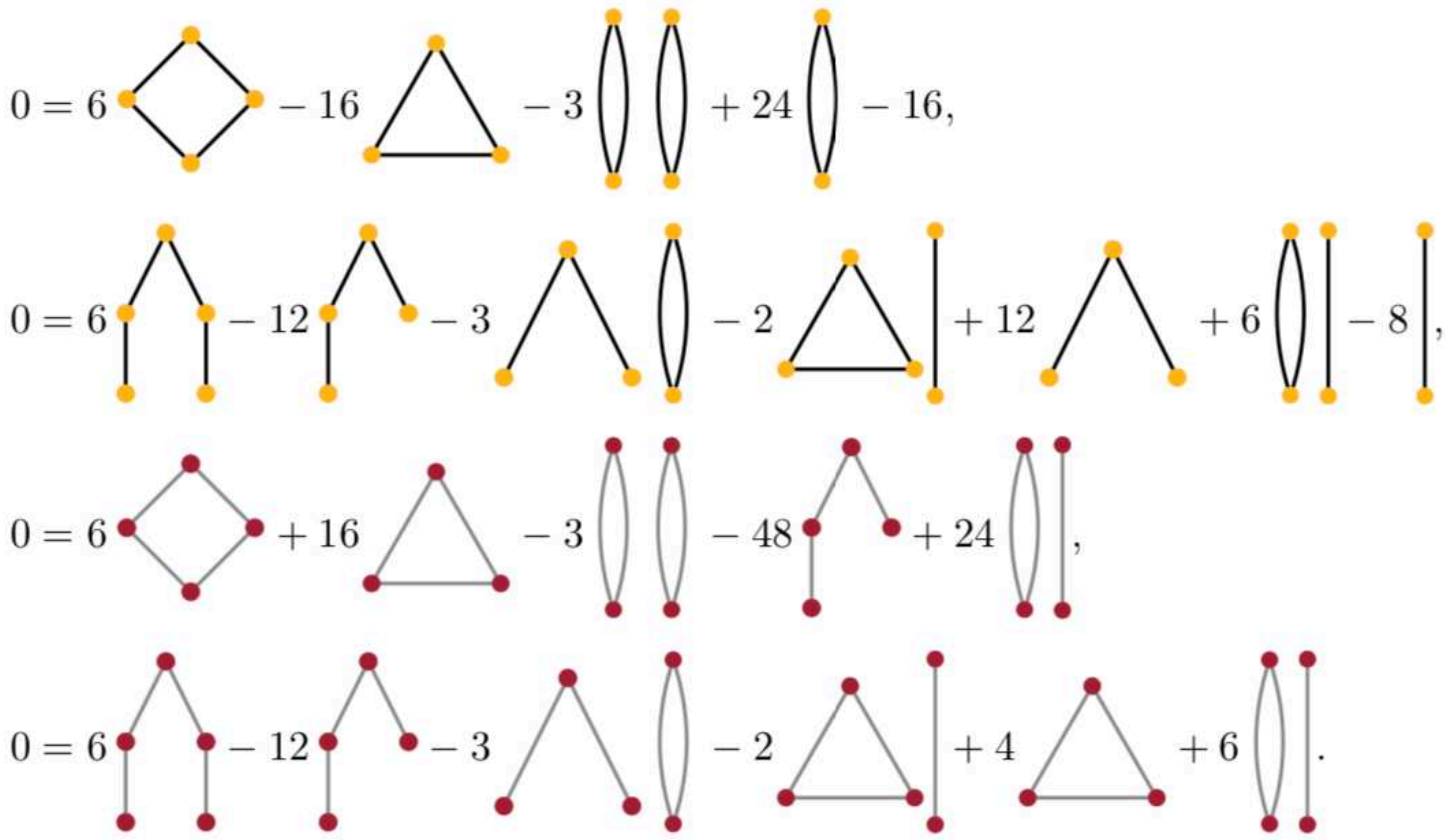
Two-Prong Classification with Varying β



$\beta = 2$ for both D_2 and C_2 for both Pythia 8 and Herwig++ works better than $\beta = 1$ for Z vs. QCD

[Larkoski, Moult, Neill, 1409.6298]

Table of Euclidean-Lorentz Identities

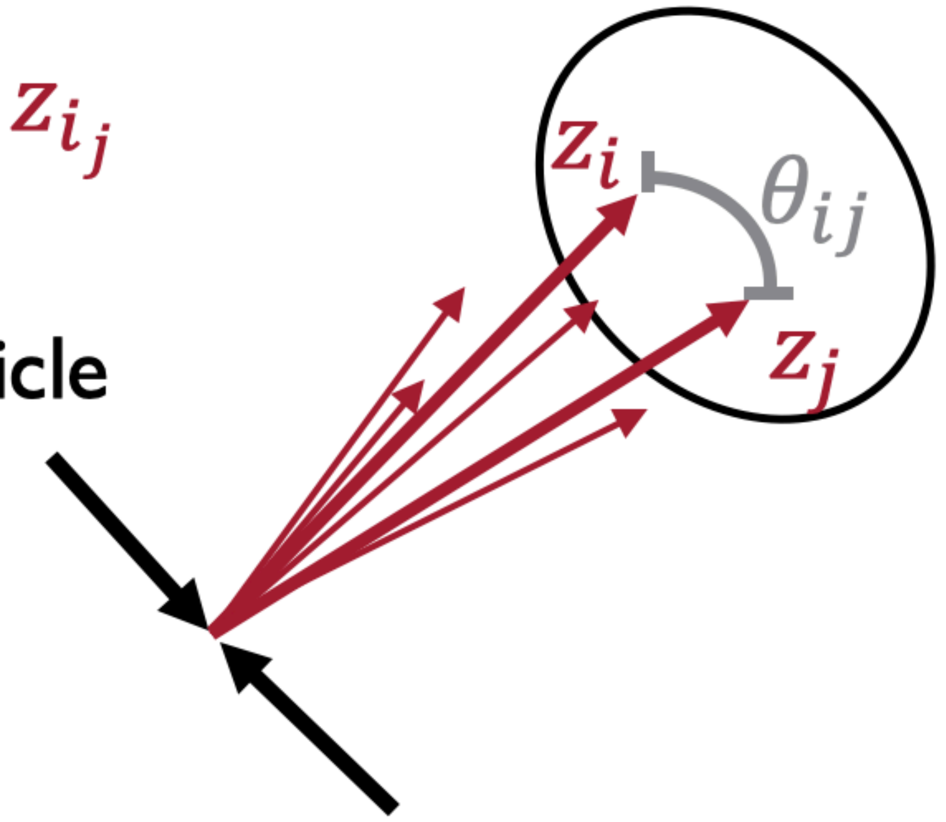


	$d=0$	$d=1$	$d=2$			$d=3$								…
	1													
	2	−1												
	4	−4	1											
	4	−4		1										
	4	−4			1									
	8	−12	6			−1								
	8	−12		6			−1							
	8	−12	2	4				−1						
	8	−12		6					−1					
	8	−12		4	2					−1				
	8	−12	2		4						−1			
	8	−12		2	4							−1		
	8	−12			6								−1	
	16	−32	24			−8								1
	16	−32	8	16				−8						
	16	−32	4	20			−4	−4						
	16	−32	12	12		−2		−6						
	16	−32		16	8					−8				
	16	−32		20	4		−2		−2	−4				
	16	−32	4	16	4			−4		−4				
	16	−32	4	12	8			−2		−4	−2			
	16	−32	4	20				−4	−4					
	16	−32		24					−8					
	16	−32		16	8				−2	−4		−2		
	16	−32		12	12					−4		−4		
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱

Derivation of EFP Linear Spanning Basis

Arbitrary IRC-safe observable: $S(p_1^\mu, \ldots, p_M^\mu)$

- Energy expansion*: Approximate S with polynomials of z_{i_j}
  - IR safety: S is unchanged under addition of soft particle
  - C safety: S is unchanged under collinear splitting of a particle
  - Relabeling symmetry: Particle index is arbitrary



[F. Tkachov, hep-ph/9601308]

Energy correlator parametrized by angular function f

$$\sum_{i_1=1}^{M} \cdots \sum_{i_N=1}^{M} z_{i_1} \ldots z_{i_N} f(\hat{p}_{i_1}, \ldots, \hat{p}_{i_N})$$

➡ Energy correlators linearly span IRC-safe observables

- Angular expansion*: Approximate f with polynomials in θ_{ij}
- Simplify: Identify unique analytic structure that emerge

➡ Linear spanning basis in terms of "EFPs" has been found!

$$S \simeq \sum_{g \in G} s_G \mathrm{EFP}_G\,, \qquad \mathrm{EFP}_G \equiv \sum_{i_1=1}^{M} \cdots \sum_{i_N=1}^{M} z_{i_1} \ldots z_{i_N} \prod_{(k,\ell)\in G} \theta_{i_k i_\ell}$$

**Generically, approximations exist by the Stone-Weierstrass theorem

Rewriting General EFP as Contraction of EFMs

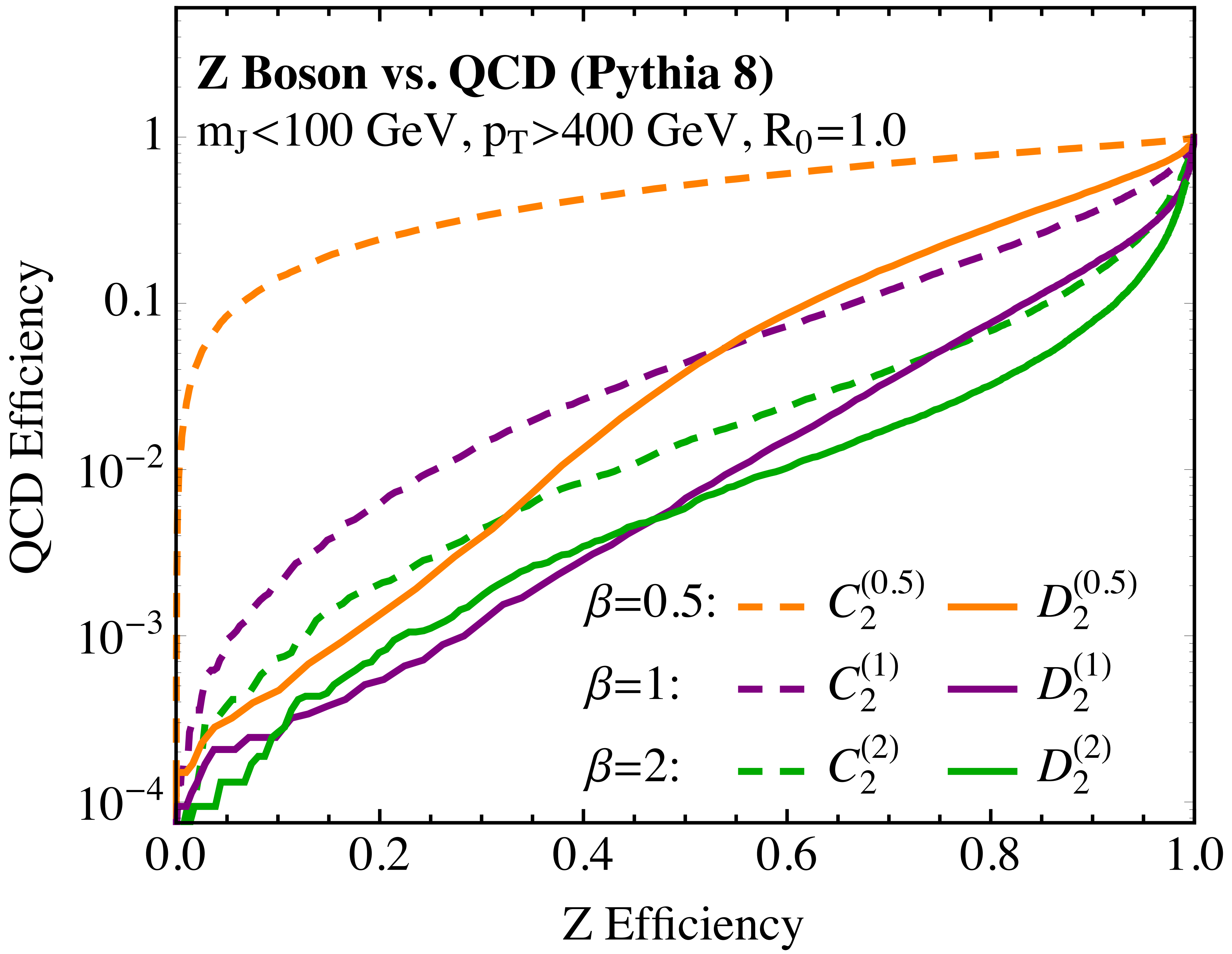
$$\begin{aligned}
\mathrm{EFP}_G &= \sum_{i_1=1}^{M} \cdots \sum_{i_N=1}^{M} E_{i_1} \cdots E_{i_N} \prod_{(k,\ell)\in G} 2\eta_{\mu\nu} n_{i_k}^{\mu} n_{i_\ell}^{\nu} \\
&= \left(\prod_{j=1}^{N} \sum_{i_j=1}^{M} z_{i_j} n_{i_j}^{\mu_1^j} n_{i_j}^{\mu_2^j} \cdots n_{i_j}^{\mu_{v_j}^j} \right) \prod_{(k,\ell)\in G} 2\eta_{\mu^k_{A_{k\ell}} \mu^\ell_{A_{\ell k}}} \\
&= \left(\prod_{j=1}^{N} \mathcal{I}^{\mu_1^j \mu_2^j \cdots \mu_{v_j}^j} \right) \prod_{(k,\ell)\in G} 2\eta_{\mu^k_{A_{k\ell}} \mu^\ell_{A_{\ell k}}},
\end{aligned}$$

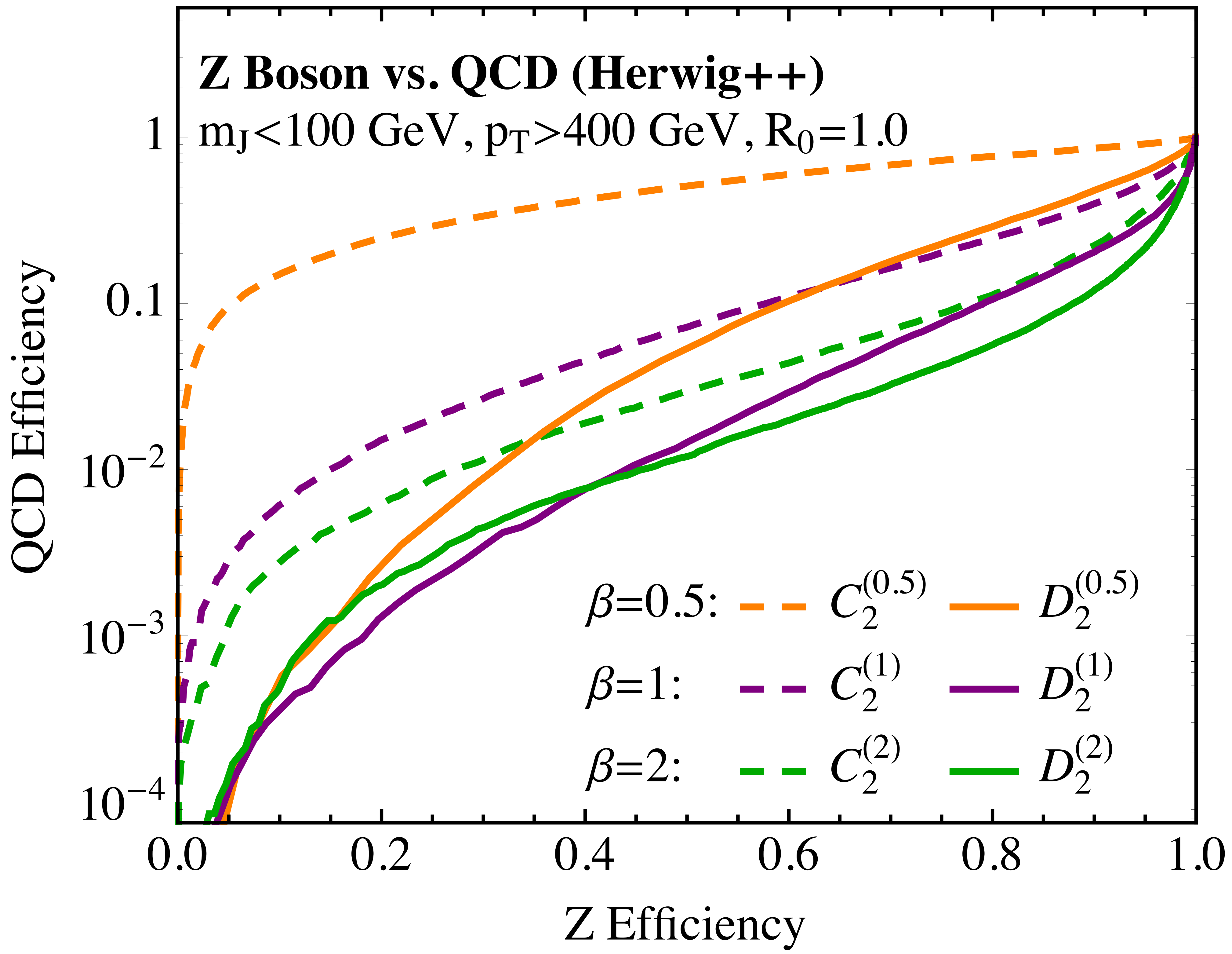
Graph adjacency matrix $A_{\ell k}$

EFM

Graph-specific contraction

Two-Prong Classification with Varying β



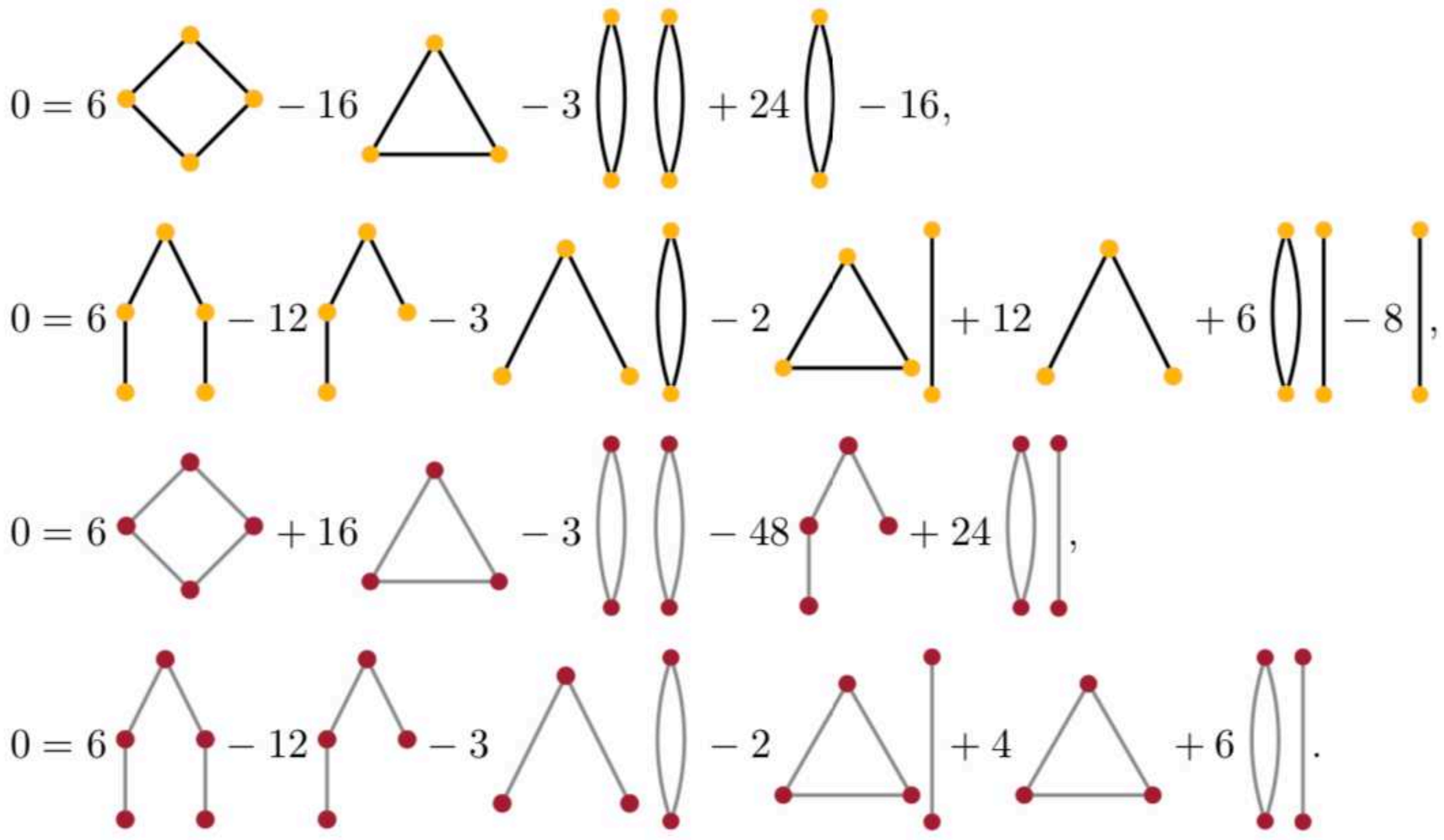


$\beta = 2$ for both D_2 and C_2 for both Pythia 8 and Herwig++ works better than $\beta = 1$ for Z vs. QCD

[Larkoski, Moult, Neill, 1409.6298]

Table of Euclidean-Lorentz Identities

Valid only in e^+e^- collisions



	$d=0$	$d=1$	$d=2$			$d=3$								…
	1													
	2	−1												
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	4	−4			1									
	8	−12	6			−1								
	8	−12		6			−1							
	8	−12	2	4				−1						
	8	−12		6					−1					
	8	−12		4	2					−1				
	8	−12	2		4						−1			
	8	−12		2	4							−1		
	8	−12			6								−1	
	16	−32	24			−8								1
	16	−32	8	16				−8						
	16	−32	4	20			−4	−4						
	16	−32	12	12		−2		−6						
	16	−32		16	8					−8				
	16	−32		20	4		−2		−2	−4				
	16	−32	4	16	4			−4		−4				
	16	−32	4	12	8			−2		−4	−2			
	16	−32	4	20				−4	−4					
	16	−32		24					−8					
	16	−32		16	8				−2	−4		−2		
	16	−32		12	12					−4		−4		
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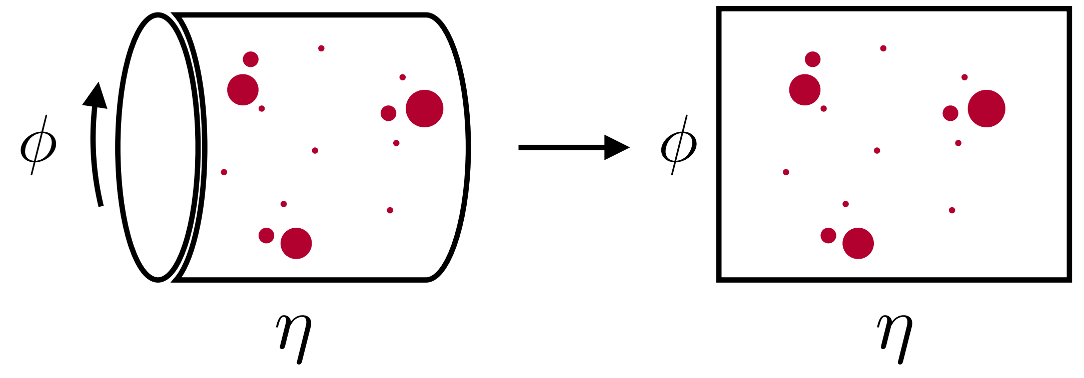
Explicit Geometry – Events as Distributions of Energy



[PTK, Metodiev, Thaler, JHEP 2019; PTK, Metodiev, Thaler, JHEP 2020]

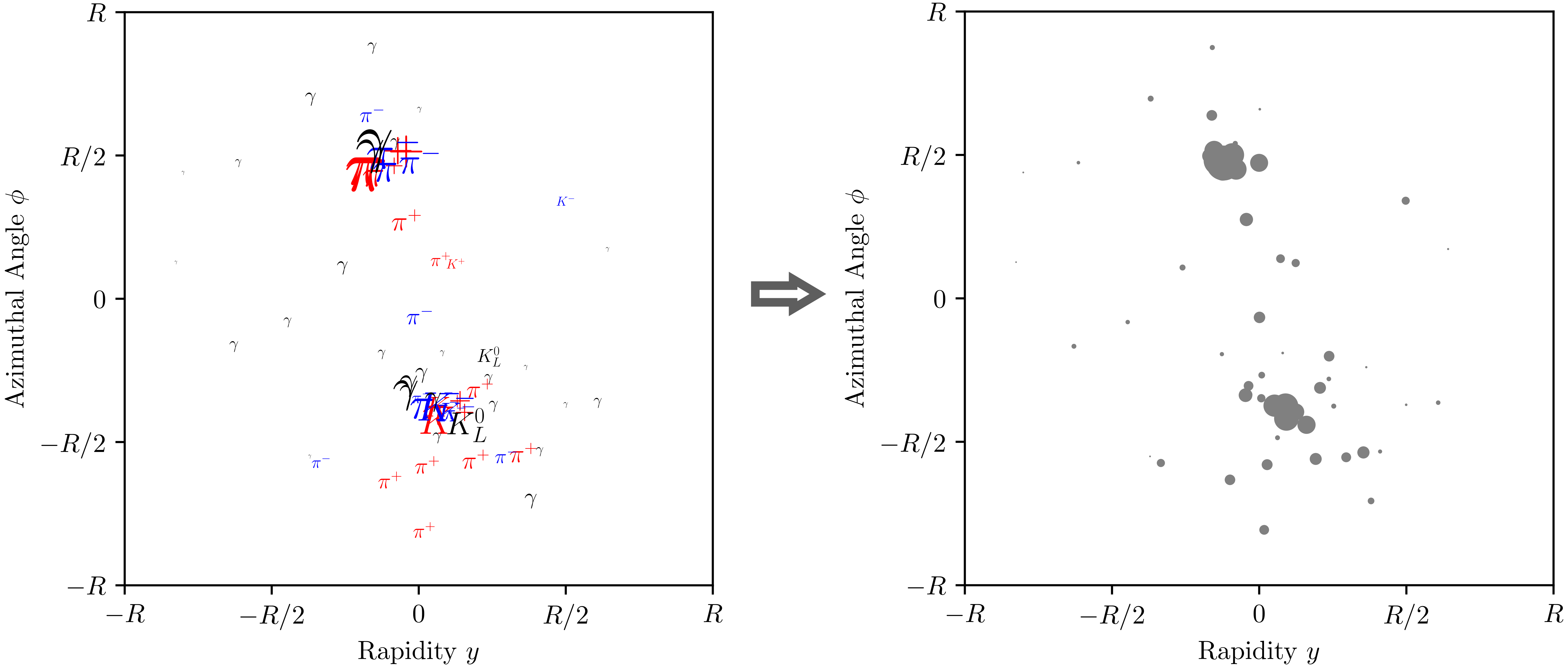
Energy flow distribution fully captures IRC-safe information

$$\mathcal{E}(\hat{n}) = \sum_{i=1}^{M} E_i \, \delta(\hat{n} - \hat{n}_i)$$

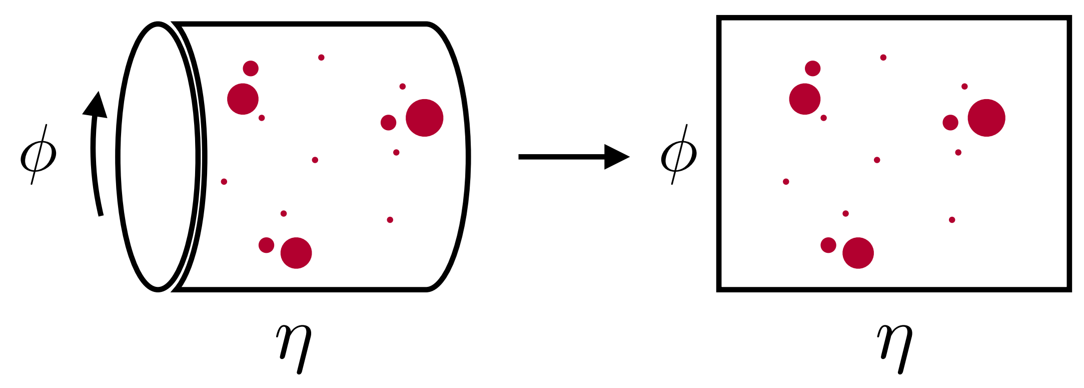
Energy Flow Distribution | Energy (p_T) | Direction (y,φ)

Full event is a set of particles having momentum and charge/flavor

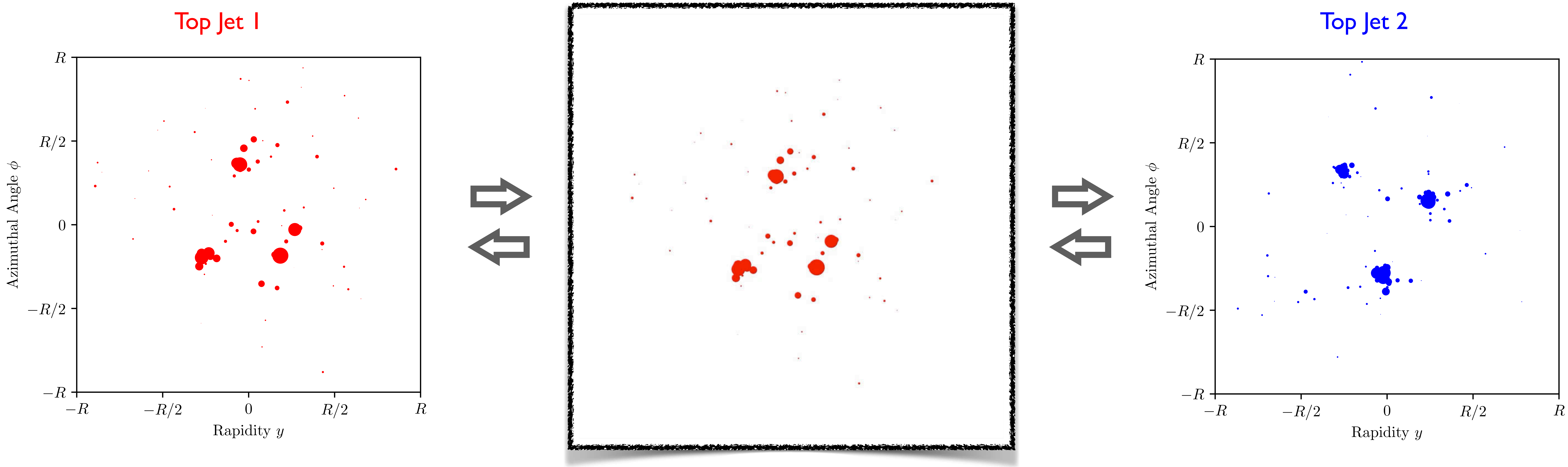
The energy flow is unpixelized and ignores charge/flavor information



Towards a Hidden Geometry – When are two events similar?



Optimal transport minimizes the "work" (stuff x distance) required to transport supply to demand



$$\mathcal{E}(\hat{n}) = \sum_{i=1}^{M} E_i\,\delta(\hat{n} - \hat{n}_i)$$

Provides a metric on normalized distributions in a space with a ground distance measure

↳ symmetric, non-negative, triangle inequality, zero iff identical

$$\theta_{ij} = \sqrt{\Delta y_{ij}^2 + \Delta \phi_{ij}^2}$$

[Peleg, Werman, Rom, IEEE 1989; Rubner, Tomasi, Guibas, ICCV 1998, ICJV 2000; Pele, Werman, ECCV 2008; Pele, Taskar, GSI 2013]

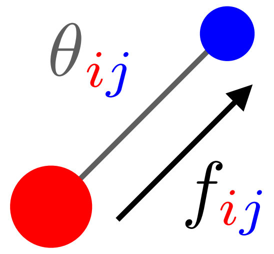
The Energy Mover's Distance (EMD)

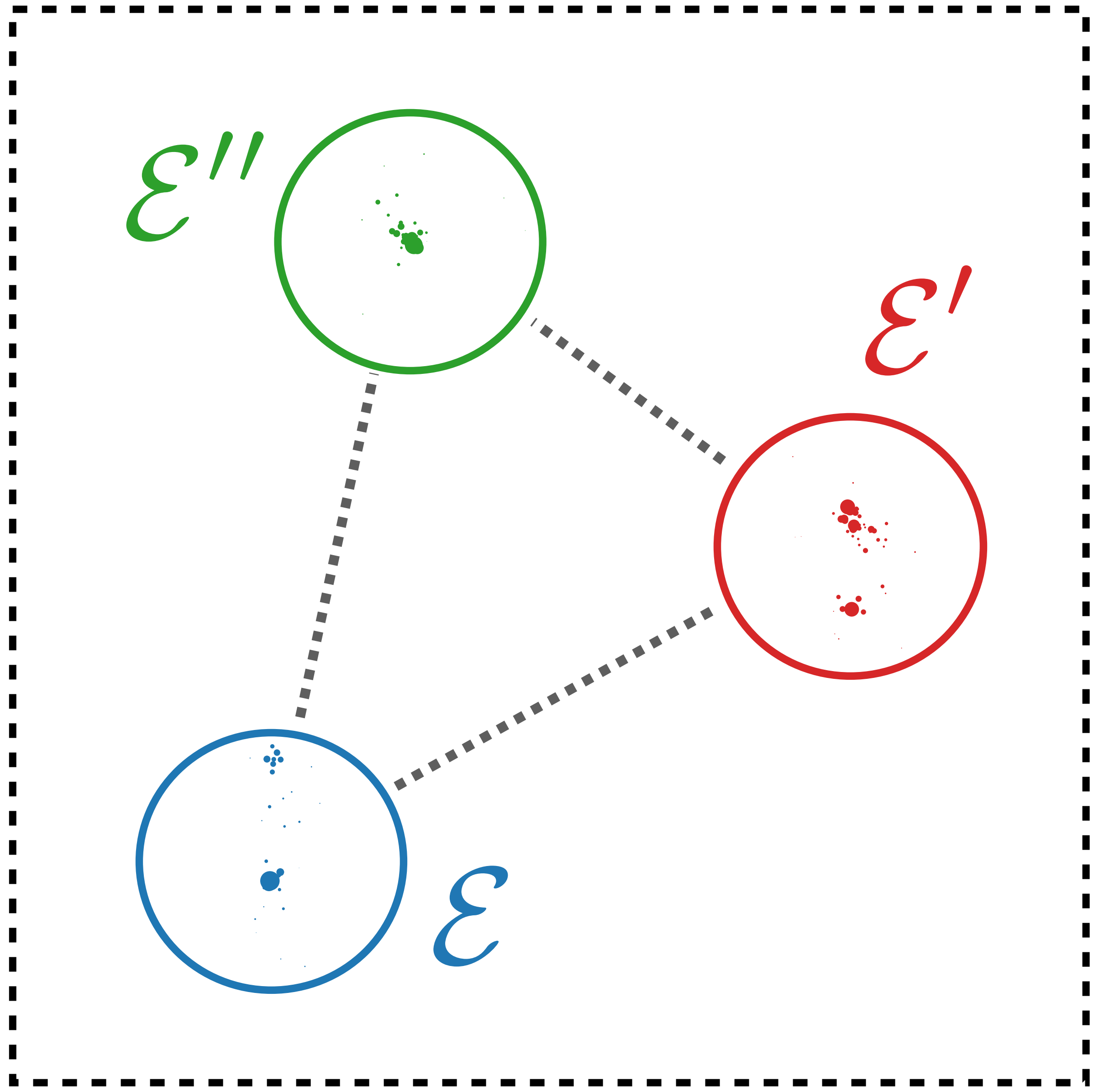
[PTK, Metodiev, Thaler, PRL 2019]

EMD between energy flows defines a metric on the space of events

$$\mathrm{EMD}_{\beta,R}(\mathcal{E},\mathcal{E}') = \underbrace{\min_{\{f_{ij}\geq 0\}} \sum_i \sum_j f_{ij} \left(\frac{\theta_{ij}}{R}\right)^{\beta}}_{\text{Cost of optimal transport}} + \underbrace{\left| \sum_i E_i - \sum_j E'_j \right|}_{\text{Cost of energy creation}}$$

$$\underbrace{\sum_j f_{ij} \leq E_i, \quad \sum_i f_{ij} \leq E'_j, \quad \sum_{ij} f_{ij} = \min\left(\sum_i E_i, \sum_j E'_j\right)}_{\text{Capacity constraints to ensure proper transport}}$$





R: controls cost of transporting energy vs. destroying/creating it

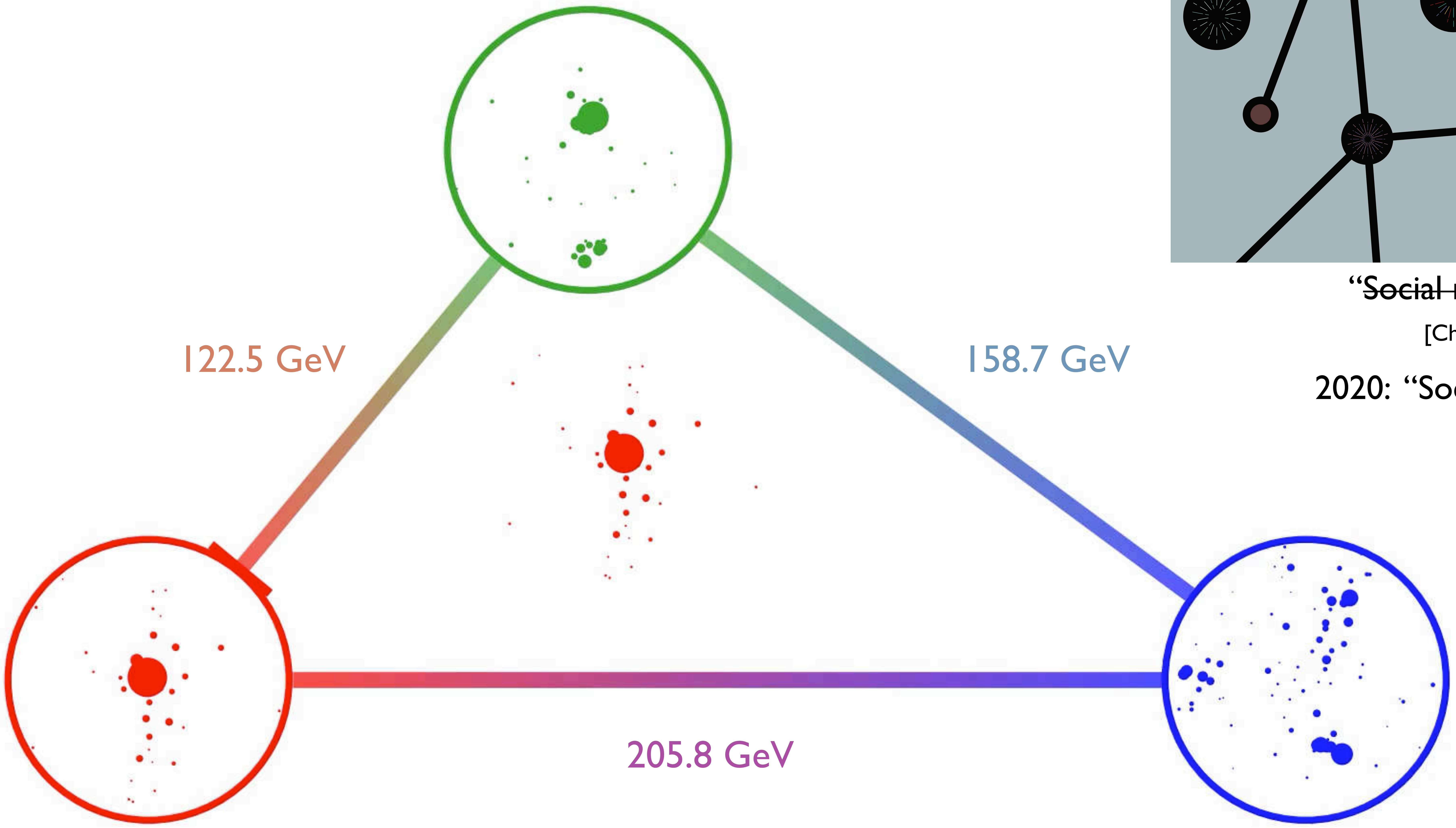
β: angular weighting exponent

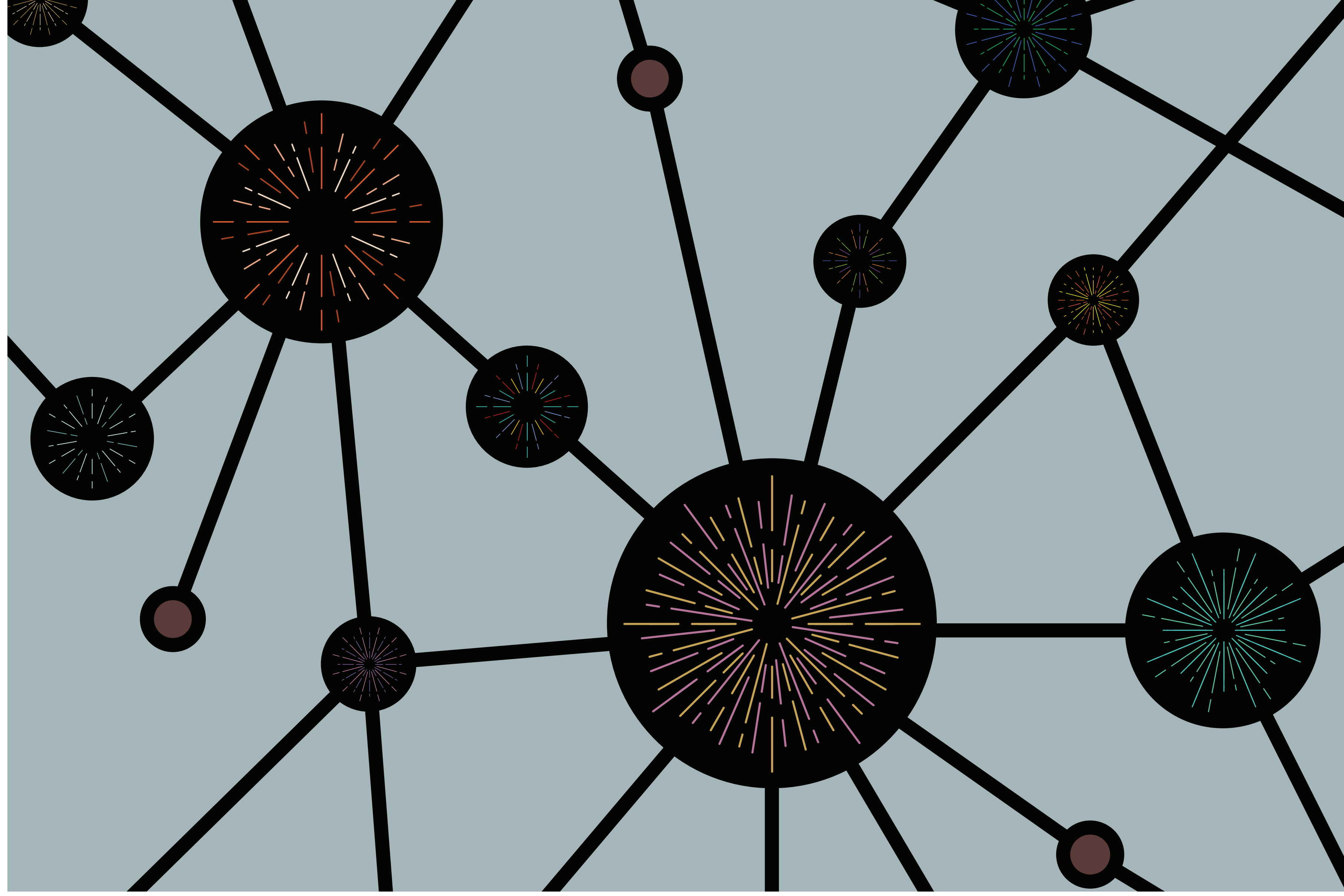
Triangle inequality satisfied for $R \geq d_{\max}/2$

$$0 \leq \mathrm{EMD}(\mathcal{E},\mathcal{E}') \leq \mathrm{EMD}(\mathcal{E},\mathcal{E}'') + \mathrm{EMD}(\mathcal{E}'',\mathcal{E}')$$

i.e. $R \geq$ jet radius for conical jets

Geodesics in the Space of Events





"~~Social networking of jets~~"

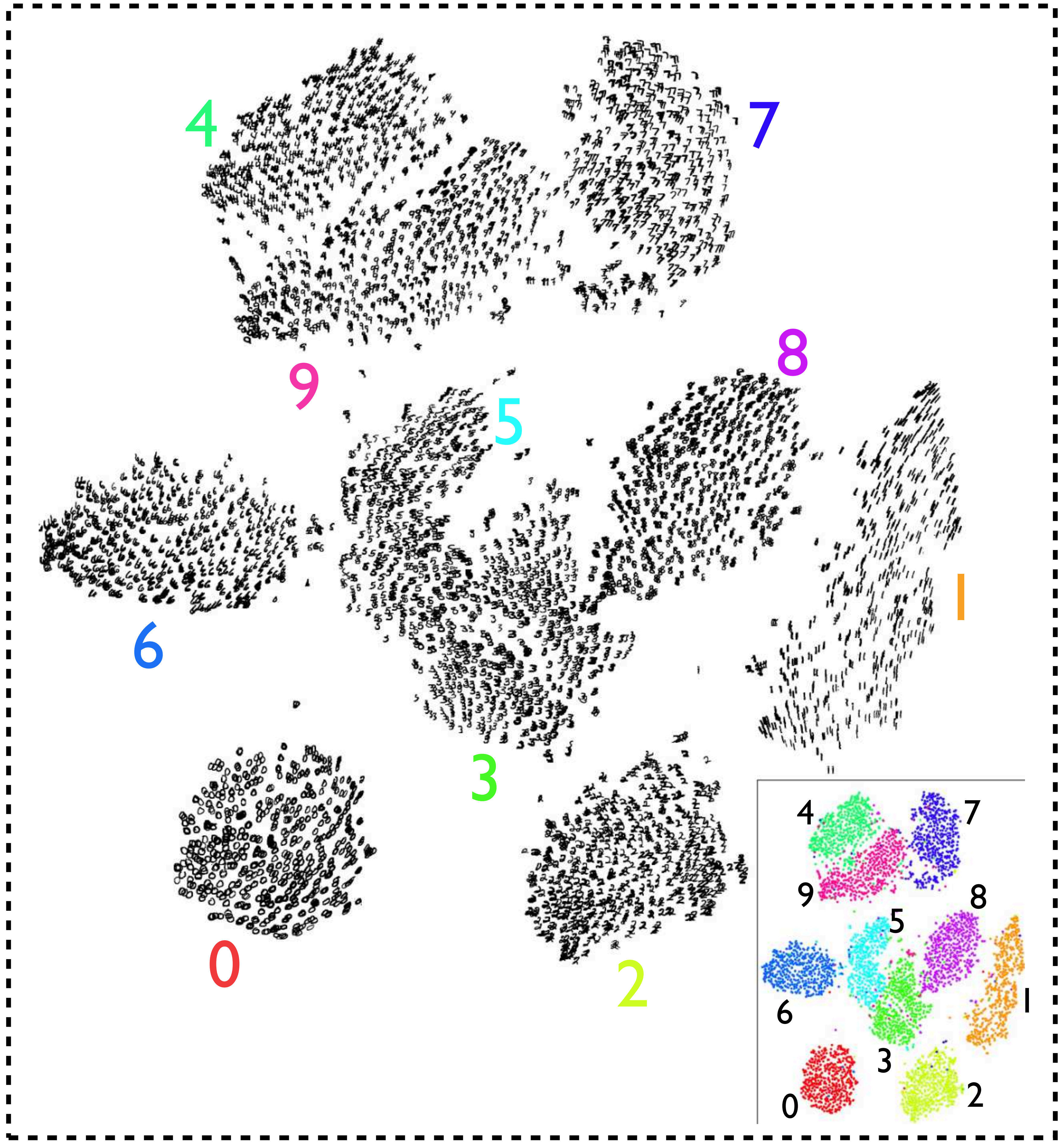
[Chu, MIT News 2019]

2020: "Social distancing of jets"

Visualizing Geometry in the Space of Events

t-Distributed Stochastic Neighbor Embedding (t-SNE)

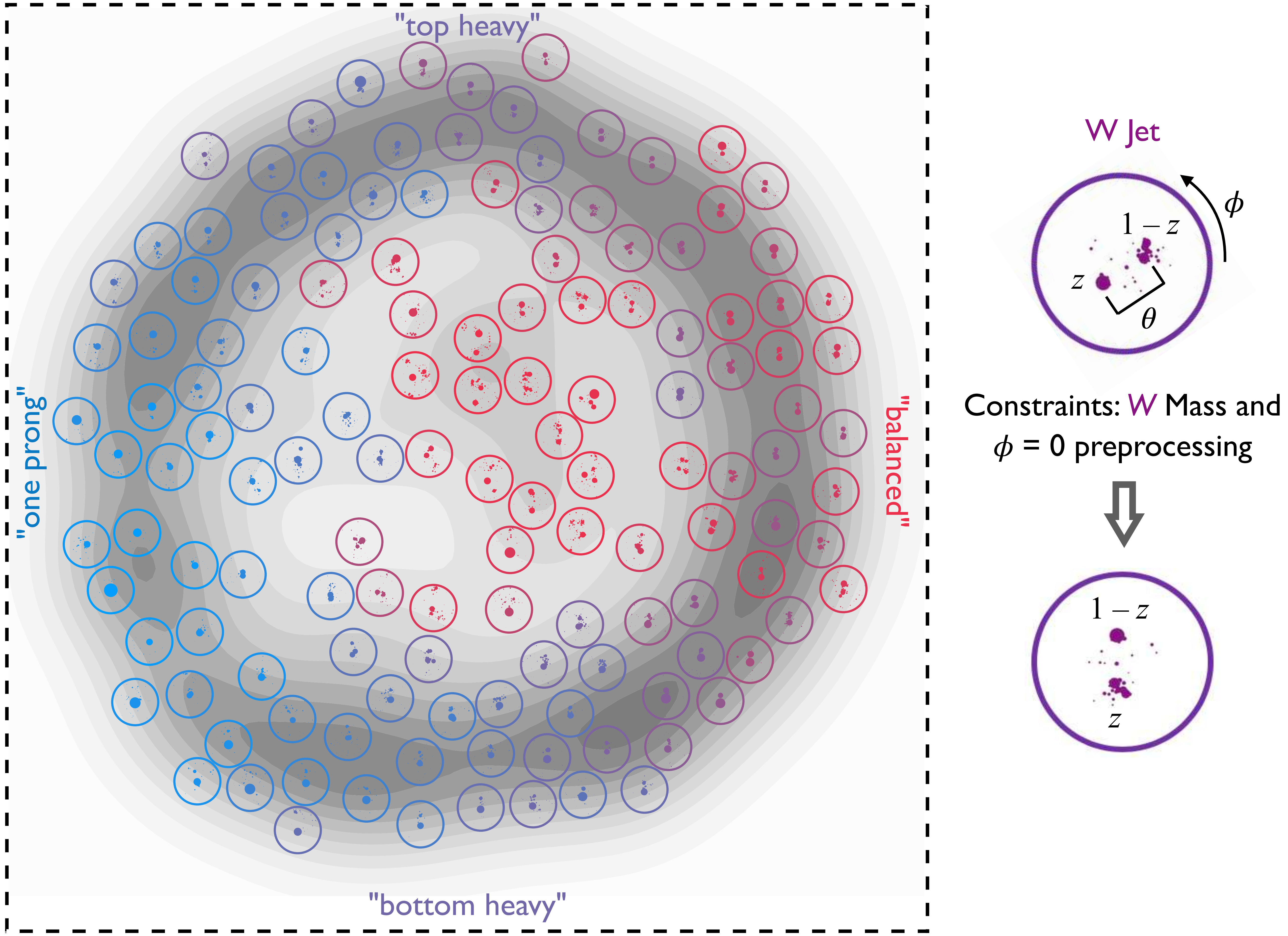
MNIST handwritten digits



[L. van der Maaten, G. Hinton, JMLR 2008]

[PTK, Metodiev, Thaler, PRL 2019]

Geometric space of W jets

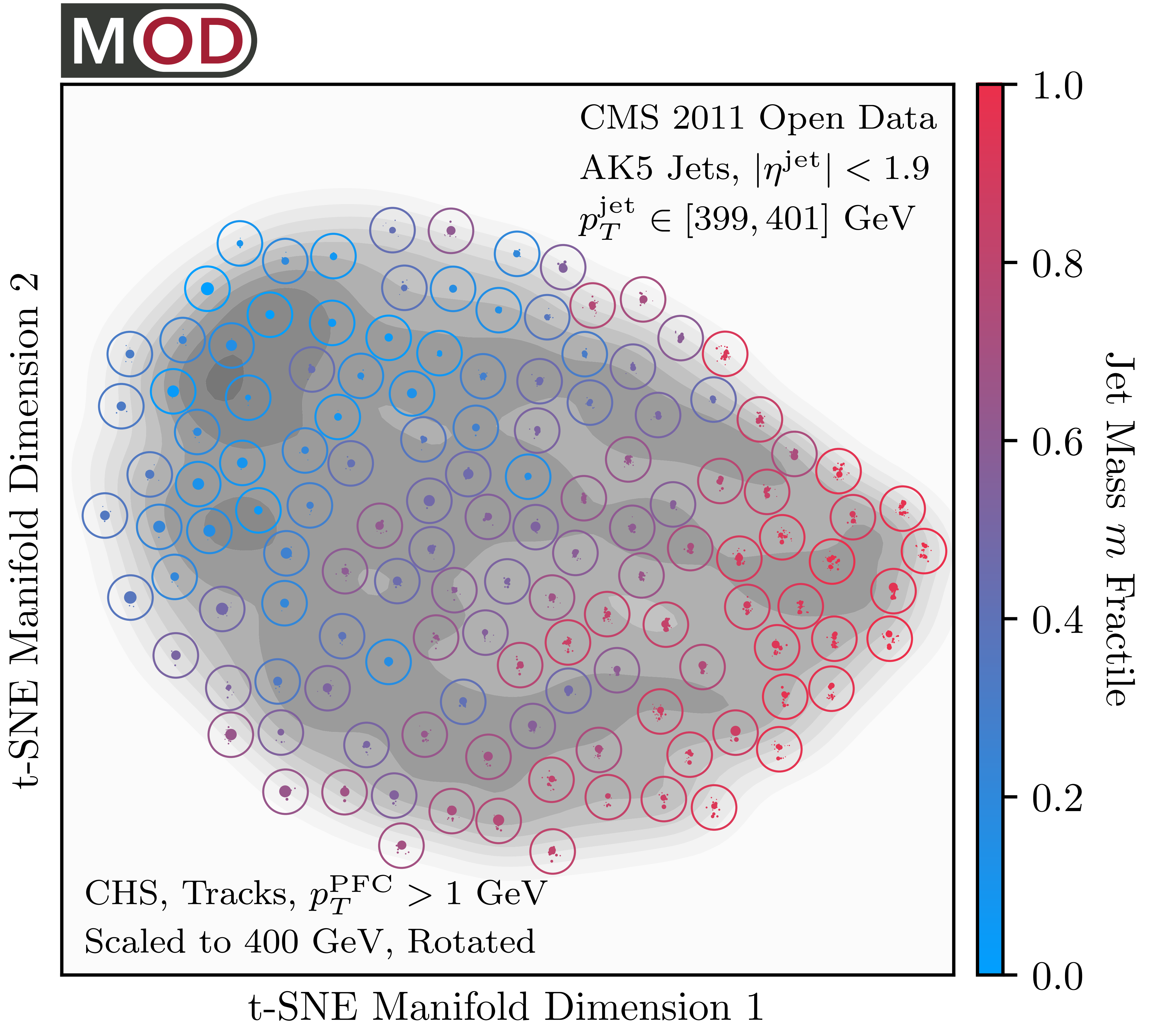


Gray contours represent the density of jets

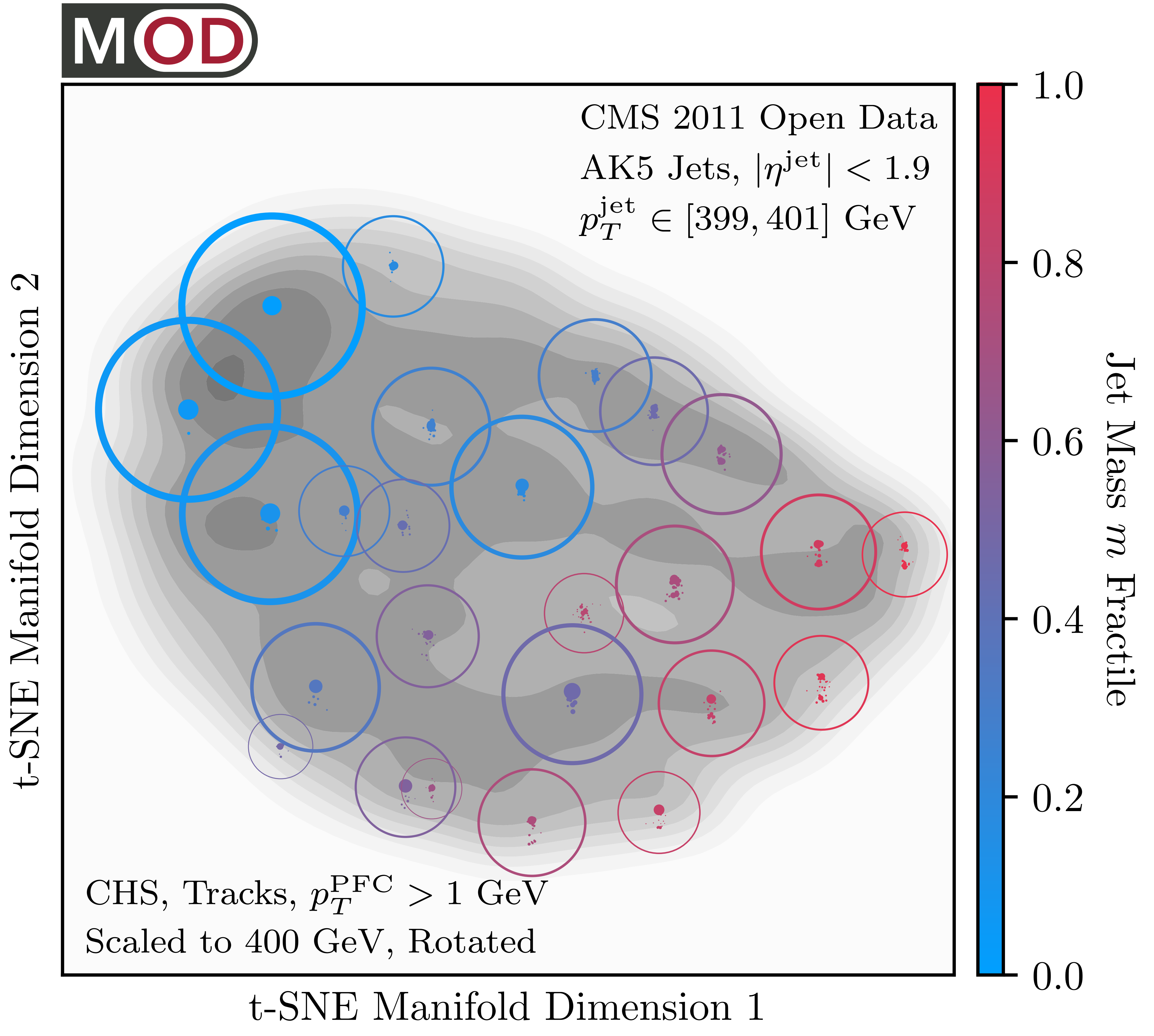
Each circle is a particular W jet

Visualizing Geometry in CMS Open Data

[PTK, Mastandrea, Metodiev, Naik, Thaler, PRD 2019; code and datasets at energyflow.network]



Example jets sprinkled throughout

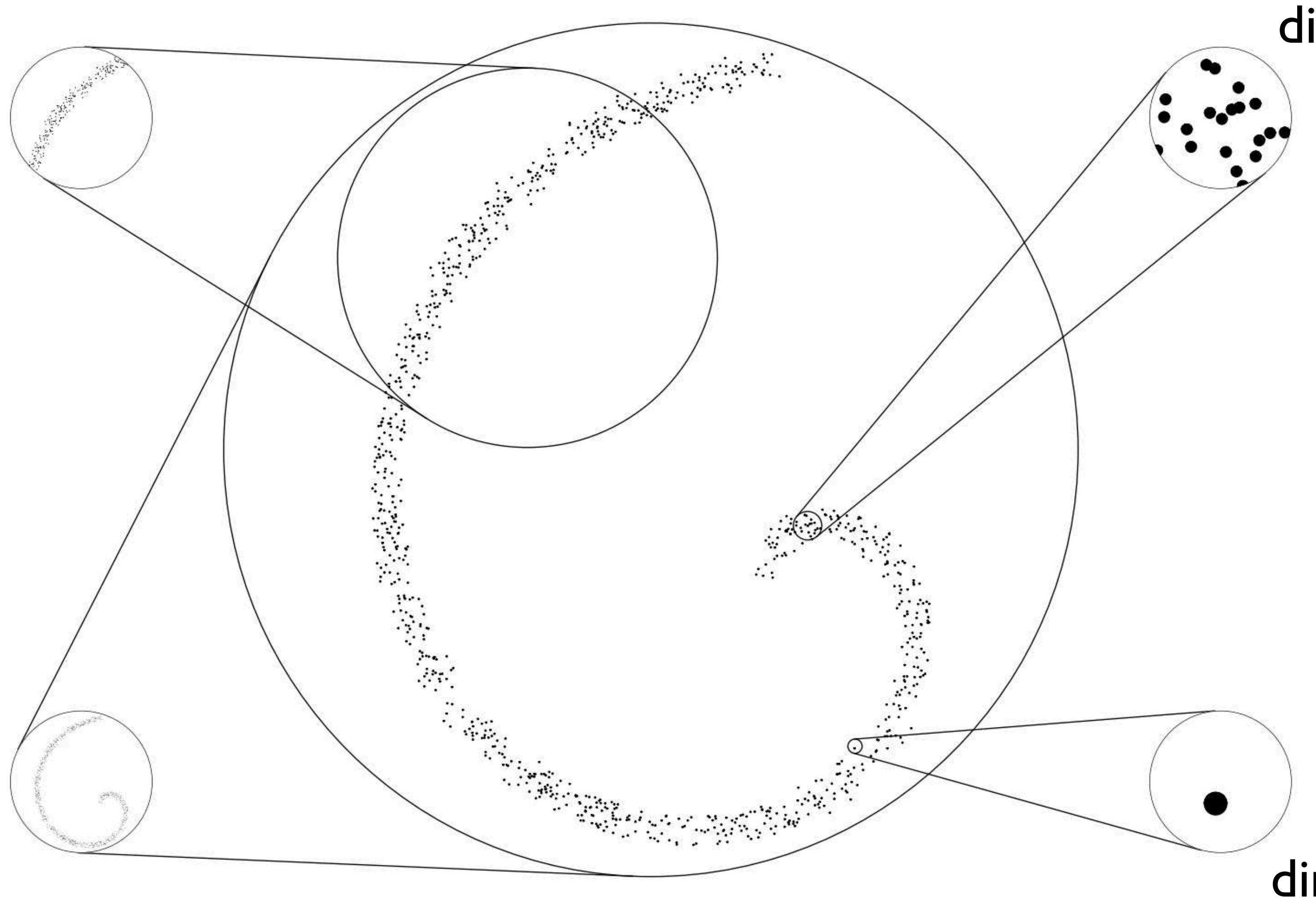


25 most representative jets ("medoids")
Size is proportional to number of jets associated to that medoid

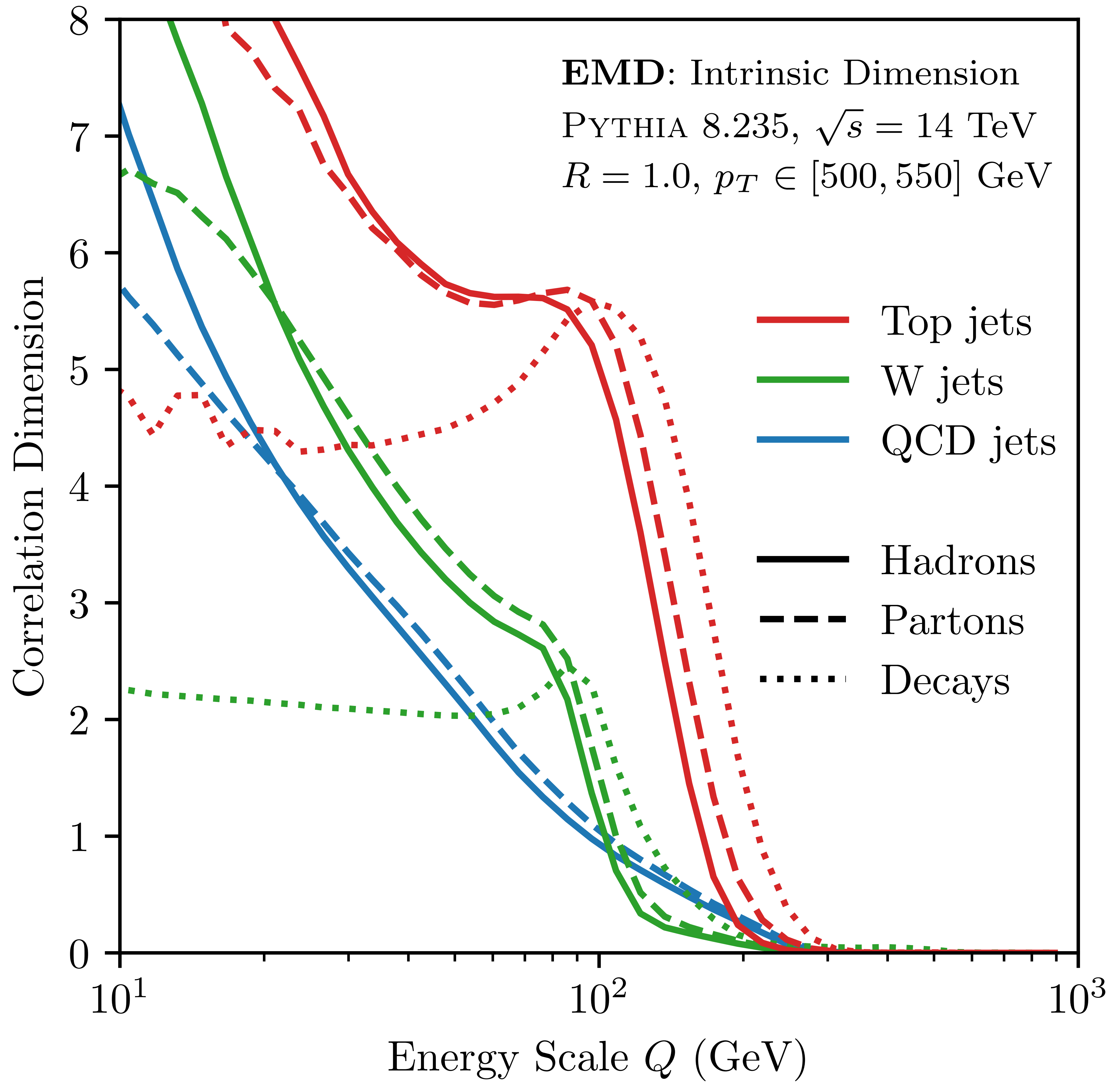
Quantifying Event-Space Manifolds

$$\dim(Q) = Q\frac{\partial}{\partial Q}\ln\sum_i\sum_j\Theta(\mathrm{EMD}(\mathcal{E}_i,\mathcal{E}'_j) < Q)$$

Correlation dimension: *how does the # of elements within a ball of size Q change?*



$$N_{\text{neigh.}}(Q) \propto Q^{\dim} \implies \dim(Q) = Q\frac{d}{dQ}\ln N_{\text{neigh.}}(Q)$$



Correlation dimension lessons:
- Decays are "constant" dim. at low Q
- Complexity hierarchy: QCD < W < Top
- Fragmentation increases dim. at smaller scales
- Hadronization important around 20-30 GeV

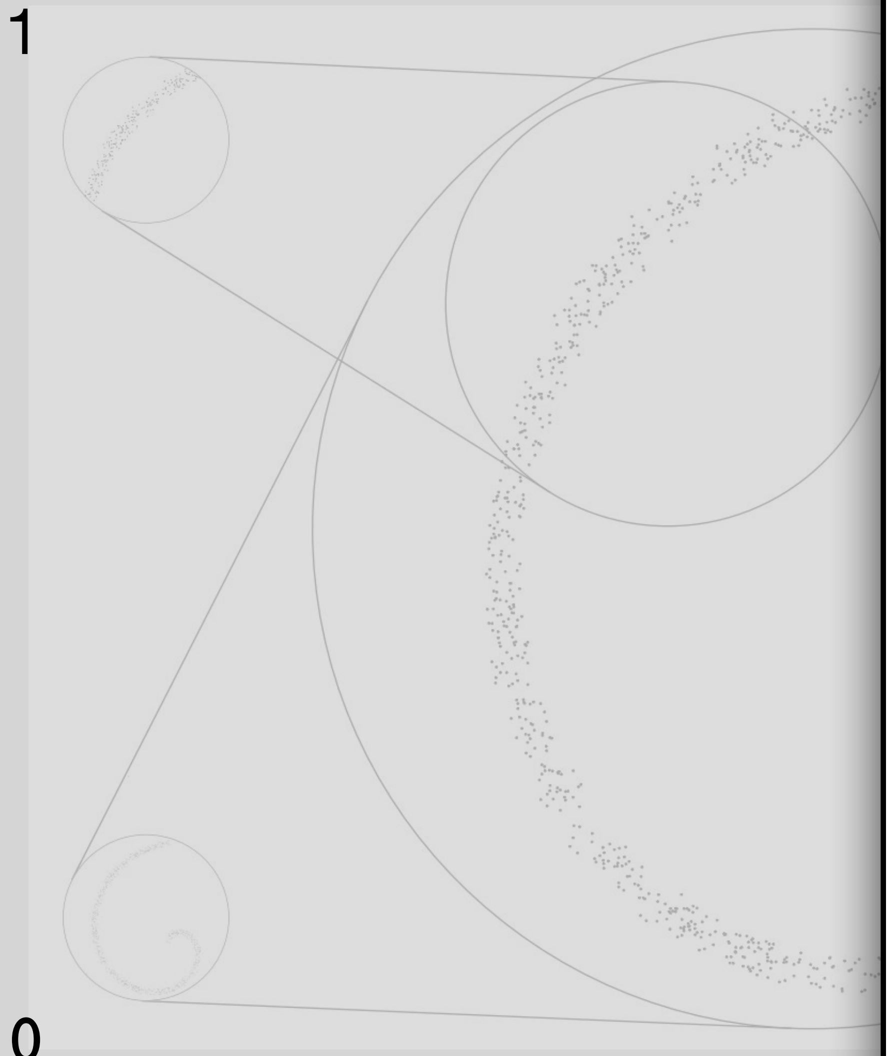
[Grassberger, Procaccia, PRL 1983; PTK, Metodiev, Thaler, PRL 2019]

*Preliminary LL calculation for QG jets in backup

Quantifying Event-Space Manifolds

Correlation dimension
elements within a ball

dim ≃ 1



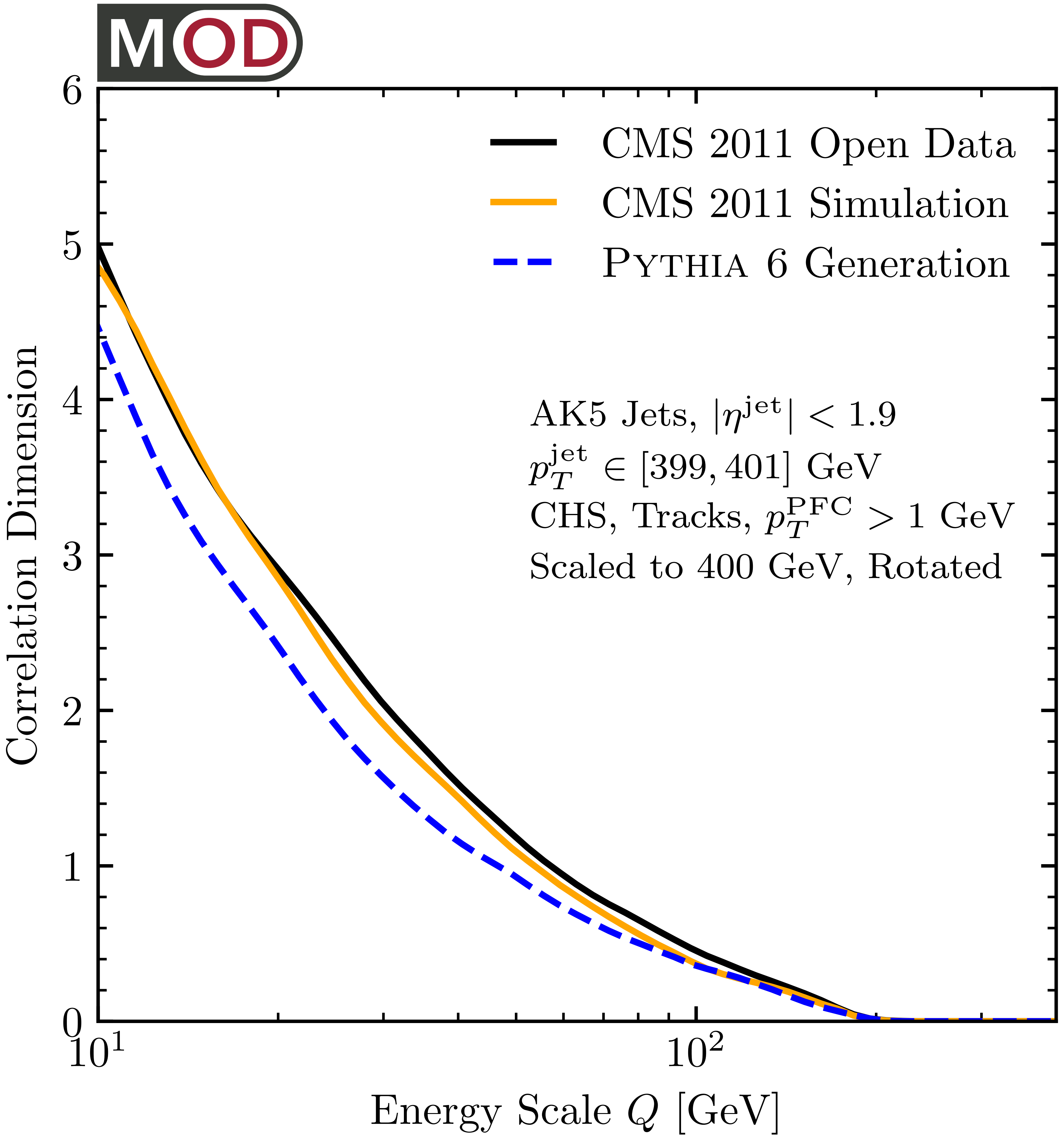
dim → 0

$N_{\text{neigh.}}(Q) \propto Q^{\text{dim}} \implies$ di

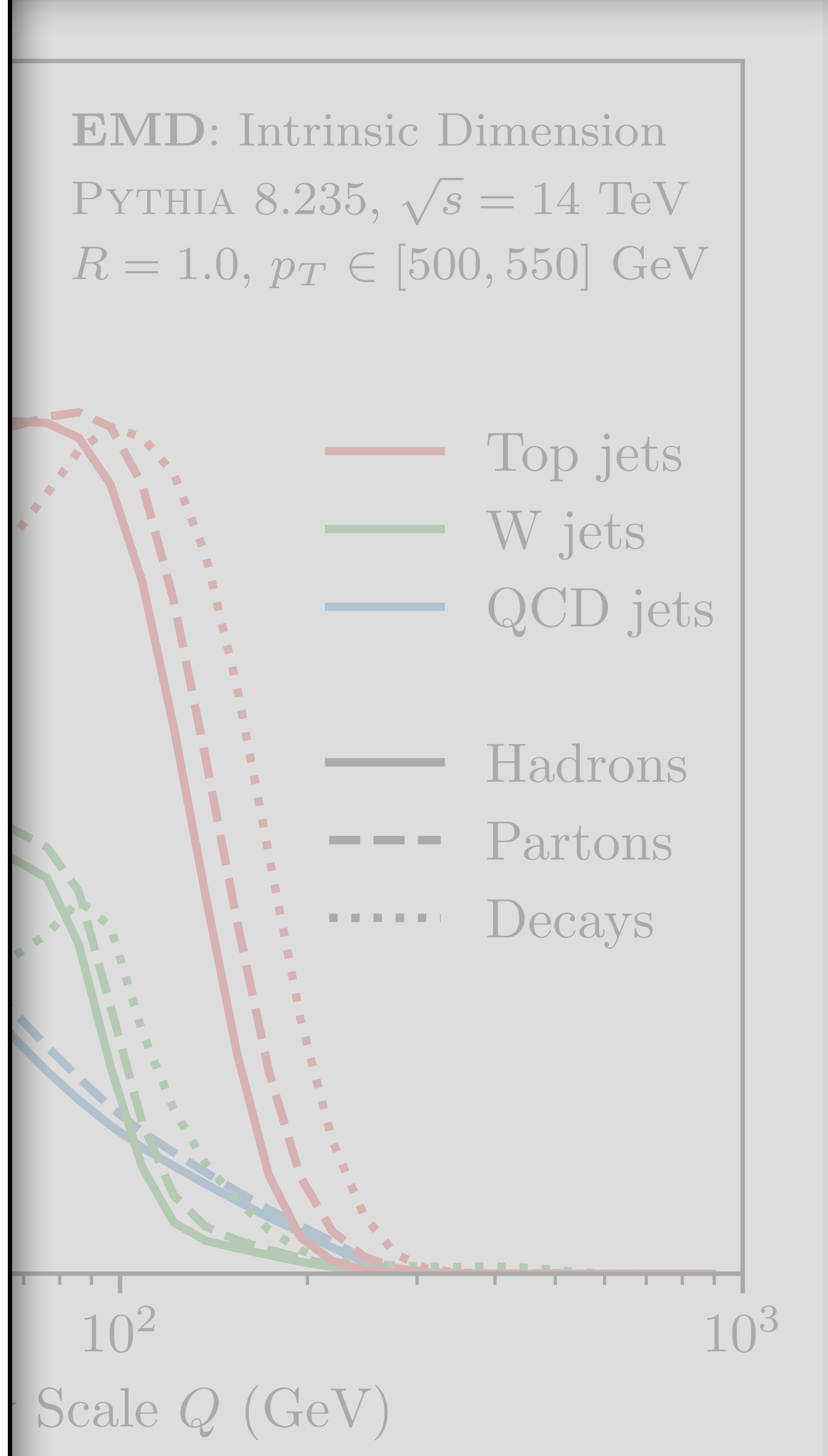
Correlation dimens
- Decays are "constan
- Complexity hierarch
- Fragmentation incre
- Hadronization impo

$\sum_j \Theta(\text{EMD}(\mathcal{E}_i, \mathcal{E}'_j) < Q)$

… in CMS Open Data



*More in backup



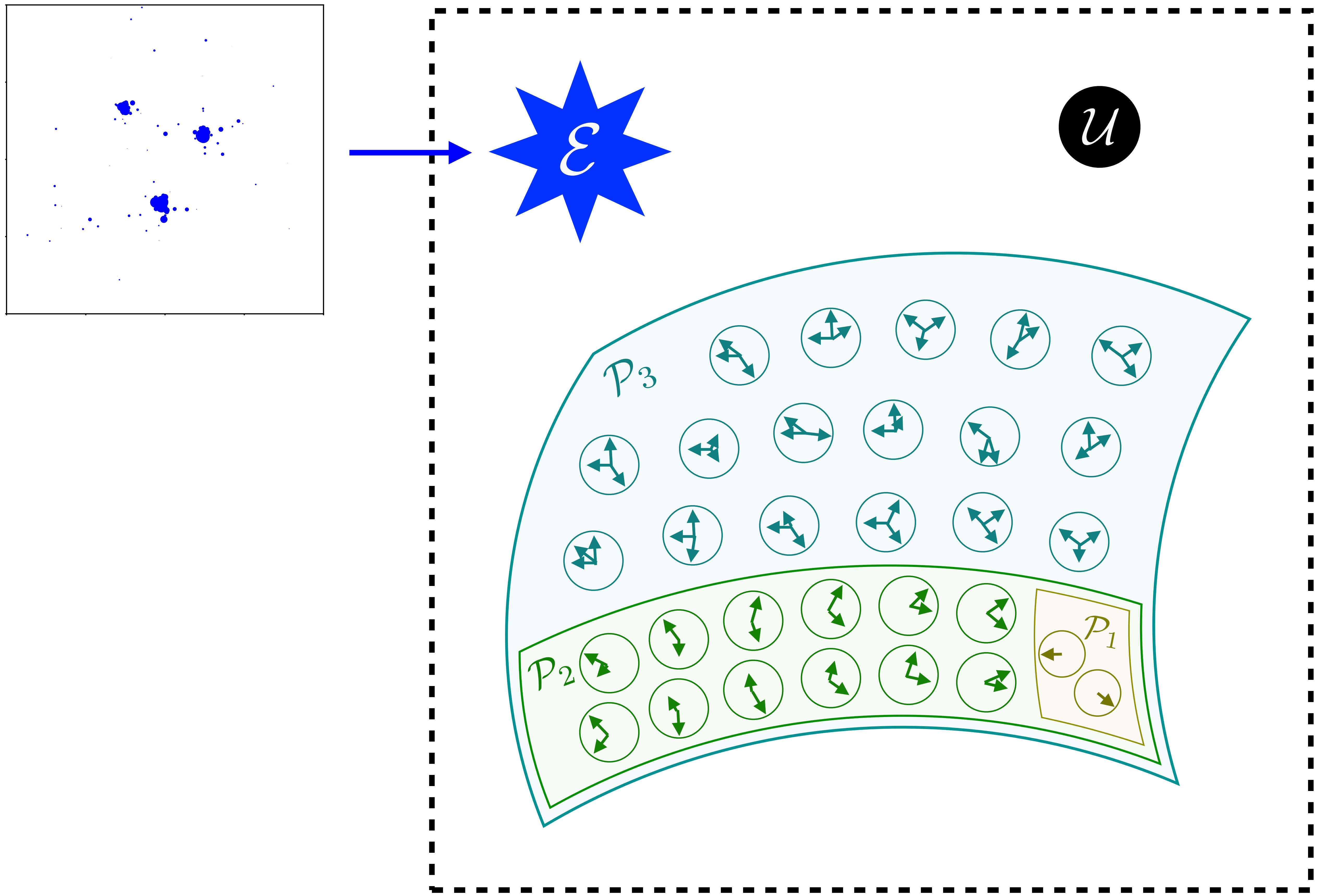
ocaccia, PRL 1983; PTK, Metodiev, Thaler, PRL 2019]

*Preliminary LL calculation for QG jets in backup

N-particle Manifolds in the Space of Events

[PTK, Metodiev, Thaler, 2004.04159]

$\mathcal{P}_N$ = set of all N-particle configurations = $\left\{ \sum_{i=1}^{N} E_i\, \delta(\hat{n} - \hat{n}_i) \,\middle|\, E_i \geq 0 \right\}$



$\mathcal{P}_1$: manifold of events with one particle

$\mathcal{P}_2$: manifold of events with two particles

$\mathcal{P}_3$: manifold of events with three particles

⋮

$\mathcal{P}_N \supset \mathcal{P}_{N-1} \supset \cdots \supset \mathcal{P}_3 \supset \mathcal{P}_2 \supset \mathcal{P}_1$

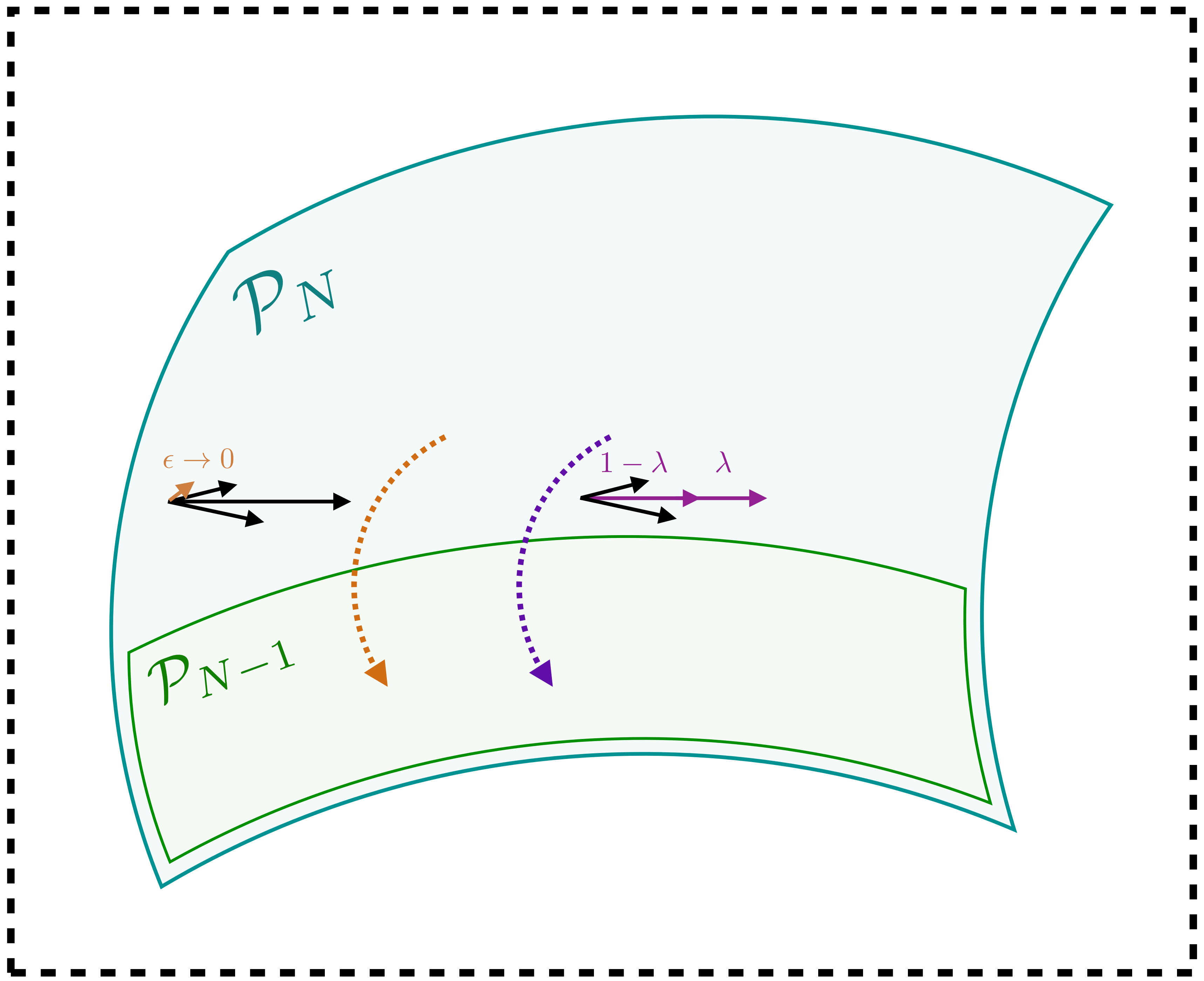
by soft and collinear limits

$\mathcal{U}$ Uniform event, not contained in any P_N

N-particle Manifolds in the Space of Events – Infrared Divergences

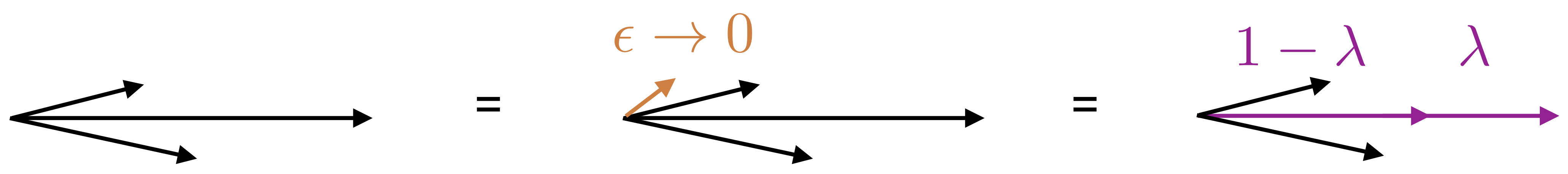
[PTK, Metodiev, Thaler, 2004.04159]

$$\mathcal{P}_N = \text{set of all N-particle configurations} = \left\{ \sum_{i=1}^{N} E_i\, \delta(\hat{n} - \hat{n}_i) \;\middle|\; E_i \geq 0 \right\}$$



$$dP_{i\to ig} \simeq \frac{2\alpha_s}{\pi} C_a \frac{d\theta}{\theta} \frac{dz}{z}$$

Energy flow is unchanged by exact soft/collinear emissions



Functions of energy flow automatically satisfy exact IRC invariance!

Real and virtual divergences appear naturally together

$$\mathcal{P}_N \supset \mathcal{P}_{N-1} \supset \cdots \supset \mathcal{P}_3 \supset \mathcal{P}_2 \supset \mathcal{P}_1$$

by soft and collinear limits

Defining IRC Safety Precisely

[Sterman, Weinberg, PRL 1997; Sterman, PRD 1978; Banfi, Salam, Zanderighi, JHEP 2005]

Infrared and collinear safety is a proxy for perturbative calculability of an observable

Exact IRC invariance

$$\mathcal{O}(p_1^\mu, \ldots, p_M^\mu) = \mathcal{O}(0p_0^\mu, p_1^\mu, \ldots, p_M^\mu)$$
$$\mathcal{O}(p_1^\mu, \ldots, p_M^\mu) = \mathcal{O}(\lambda p_1^\mu, (1-\lambda)p_1^\mu, \ldots, p_M^\mu)$$

Guarantees observable is well-defined on energy flows

Allows for pathological observables, e.g. pseudo-multiplicity

Smooth IRC invariance

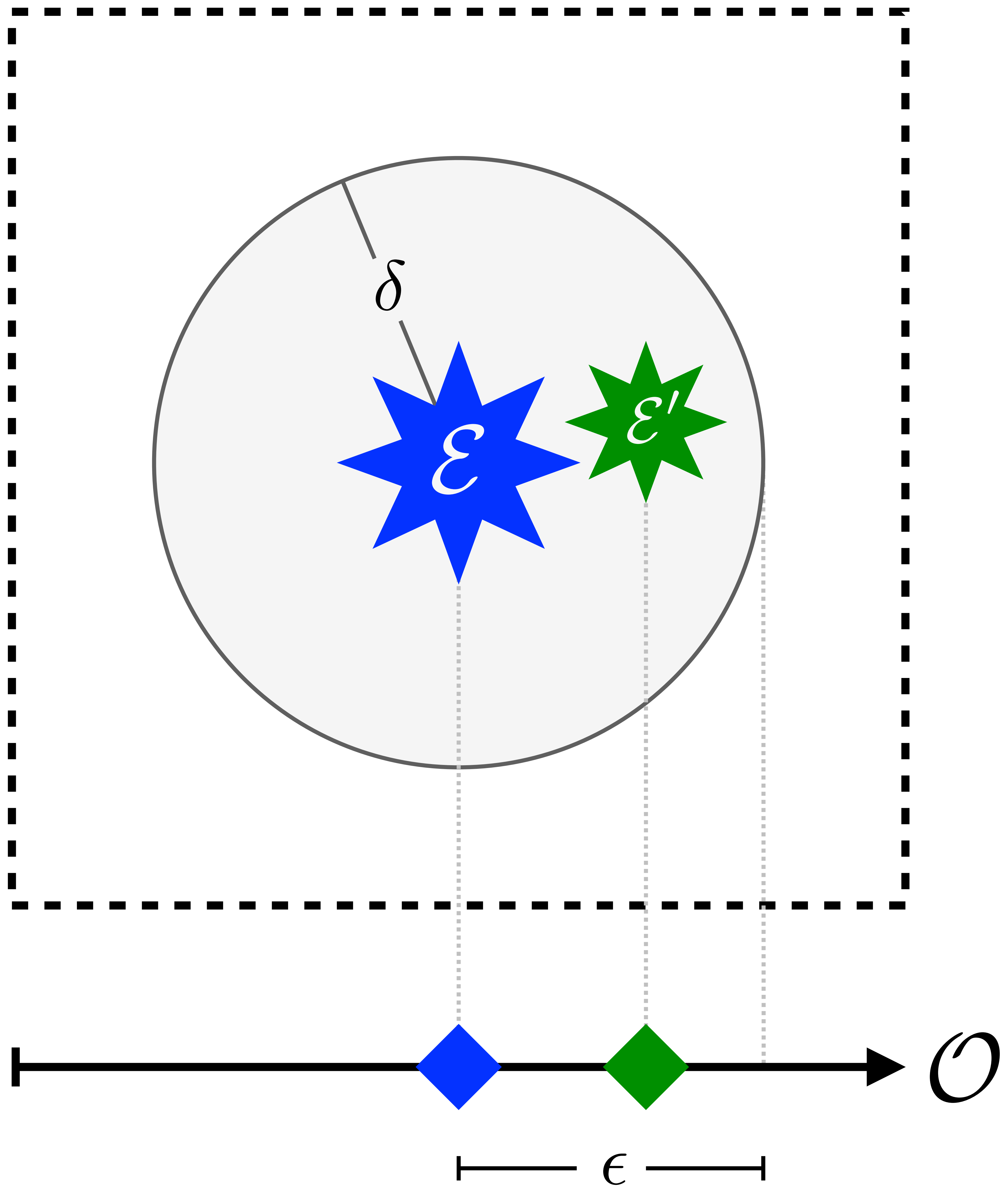
$$\mathcal{O}(p_1^\mu, \ldots, p_M^\mu) = \lim_{\epsilon\to 0} \mathcal{O}(\epsilon p_0^\mu, p_1^\mu, \ldots, p_M^\mu)$$
$$\mathcal{O}(p_1^\mu, \ldots, p_M^\mu) = \lim_{p_0^\mu \to p_1^\mu} \mathcal{O}(\lambda p_0^\mu, (1-\lambda)p_1^\mu, \ldots, p_M^\mu)$$

Eliminates common observables with hard boundaries

All Observables	Comments
Multiplicity ($\sum_i 1$)	IR unsafe and C unsafe
Momentum Dispersion [65] ($\sum_i E_i^2$)	IR safe but C unsafe
Sphericity Tensor [66] ($\sum_i p_i^\mu p_i^\nu$)	IR safe but C unsafe
Number of Non-Zero Calorimeter Deposits	C safe but IR unsafe
Defined on Energy Flows	
Pseudo-Multiplicity ($\min\{N \mid \mathcal{T}_N = 0\}$)	Robust to exact IR or C emissions
Infrared & Collinear Safe	
Jet Energy ($\sum_i E_i$)	Disc. at jet boundary
Heavy Jet Mass [67]	Disc. at hemisphere boundary
Soft-Dropped Jet Mass [38, 68]	Disc. at grooming threshold
Calorimeter Activity [69] (N_{95})	Disc. at cell boundary

More EMD Geometry – Continuity in the Space of Events

[PTK, Metodiev, Thaler, 2004.04159]



Classic $\epsilon - \delta$ definition of continuity in a metric space

An observable $\mathcal{O}$ is EMD continuous at an event $\mathcal{E}$ if, for any $\epsilon > 0$, there exists a $\delta > 0$ such that for all events $\mathcal{E}'$:

$$\mathrm{EMD}(\mathcal{E}, \mathcal{E}') < \delta \quad \implies \quad |\mathcal{O}(\mathcal{E}) - \mathcal{O}(\mathcal{E}')| < \epsilon.$$

Towards a geometric definition of IRC Safety

IRC Safety = EMD Continuity*

*on all but a negligible set‡ of events

‡a negligible set is one that contains no positive-radius EMD-ball

⋮

Perturbation Theory in the Space of Events

[PTK, Metodiev, Thaler, 2004.04159]

Sudakov safety

[Larkoski, Thaler, JHEP 2014; Larkoski, Marzani, Thaler, PRD 2015]

Some observables have discontinuities on P_N for some N

A resummed IRC-safe companion can mitigate the divergences

$$p(\mathcal{O}_{\text{Sudakov}}) = \int d\mathcal{O}_{\text{Comp.}}\, p(\mathcal{O}_{\text{Sudakov}}|\mathcal{O}_{\text{Comp.}})\, p(\mathcal{O}_{\text{Comp.}})$$

Event geometry suggests N-(sub)jettiness as universal companion

Fixed-order calculability

[Sterman, PRD 1979; Banfi, Salam, Zanderighi, JHEP 2005]

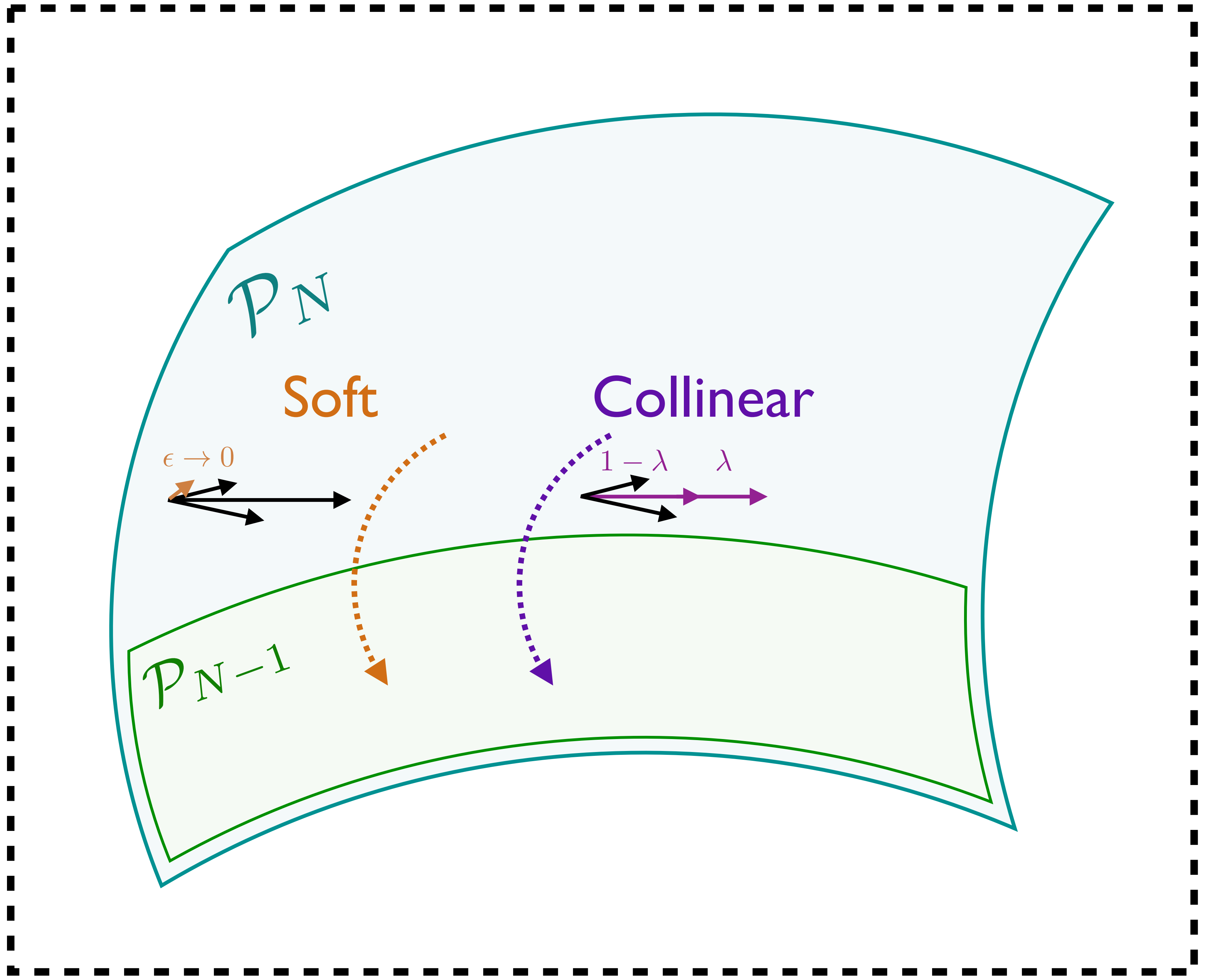
Is a statement of integrability on each P_N

EMD continuity must be upgraded to EMD-Hölder continuity on each P_N

$$\lim_{\mathcal{E}\to\mathcal{E}'} \frac{\mathcal{O}(\mathcal{E}) - \mathcal{O}(\mathcal{E}')}{\text{EMD}(\mathcal{E},\mathcal{E}')^c} = 0, \quad c > 0$$

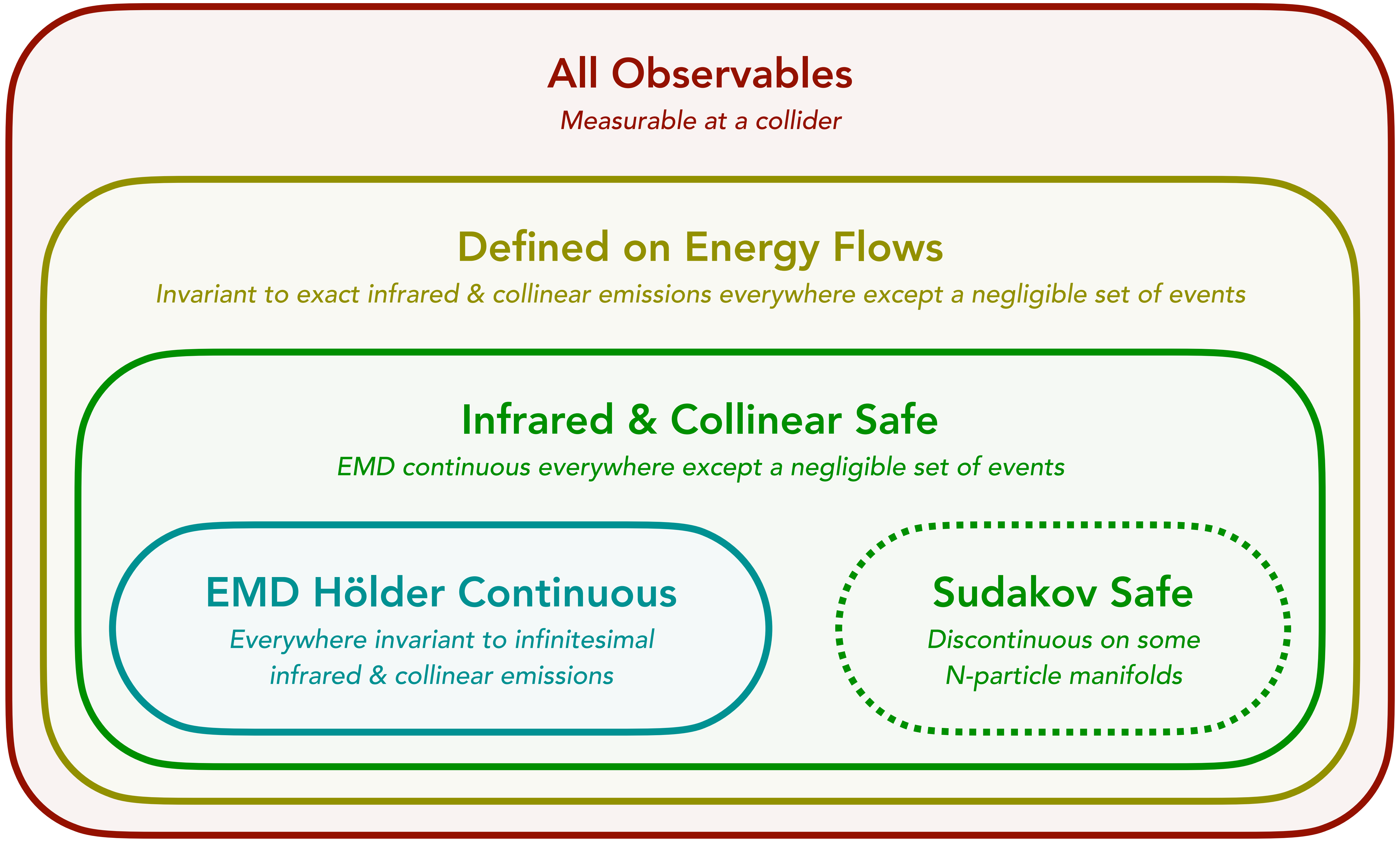
Example: $V(\mathcal{E}) = \mathcal{T}_2(\mathcal{E})\left(1 + \frac{1}{\ln E(\mathcal{E})/\mathcal{T}_3(\mathcal{E})}\right)$ is EMD continuous but not EMD Hölder continuous (it is Sudakov safe)

Infrared singularities of massless gauge theories appear on each P_N



Hierarchy of IRC Safety Definitions

[PTK, Metodiev, Thaler, 2004.04159]



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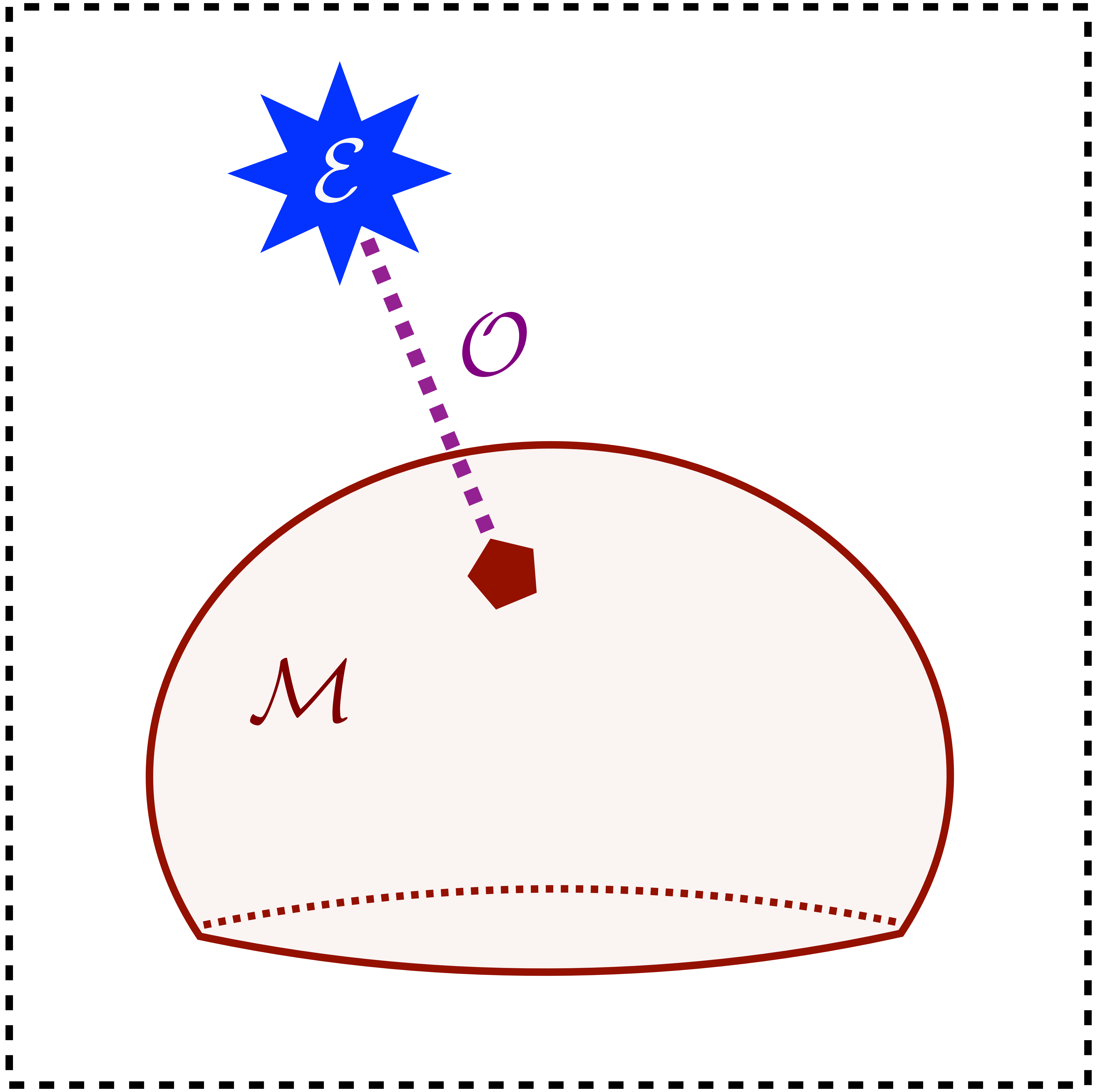
Sudakov Safe	
Groomed Momentum Fraction [39] (z_g)	Disc. on 1-particle manifold
Jet Angularity Ratios [37]	Disc. on 1-particle manifold
N-subjettiness Ratios [47, 48] (τ_{N+1}/τ_N)	Disc. on N-particle manifold
V parameter [36] (Eq. (2.11))	Hölder disc. on 3-particle manifold
EMD Hölder Continuous Everywhere	
Thrust [40, 41]	
Spherocity [42]	
Angularities [70]	
N-jettiness [44] ($\mathcal{T}_N$)	
C parameter [71–74]	Resummation beneficial at $C = \frac{3}{4}$
Linear Sphericity [72] ($\sum_i E_i n_i^\mu n_i^\nu$)	
Energy Correlators [36, 75–77]	
Energy Flow Polynomials [15, 17]	

Defining Observables via Event Space Geometry

[PTK, Metodiev, Thaler, 2004.04159]



Many common observables are distance of closest approach from event to a specific manifold

$$\boxed{\mathcal{O}(\mathcal{E}) = \min_{\mathcal{E}'\in\mathcal{M}} \mathrm{EMD}_{\beta,R}(\mathcal{E},\mathcal{E}')}$$

EMD variant for equal-energy events

$$\underbrace{\mathrm{EMD}_{\beta}(\mathcal{E},\mathcal{E}') = \lim_{R\to\infty} R^{\beta}\mathrm{EMD}_{\beta,R}(\mathcal{E},\mathcal{E}')}_{\text{Enforces equal energy (else infinity)}} \overset{\uparrow}{=} \min_{\{f_{ij}\geq 0\}} \sum_{i=1}^{M}\sum_{j=1}^{M'} f_{ij}\theta_{ij}^{\beta}$$

on equal-energy events

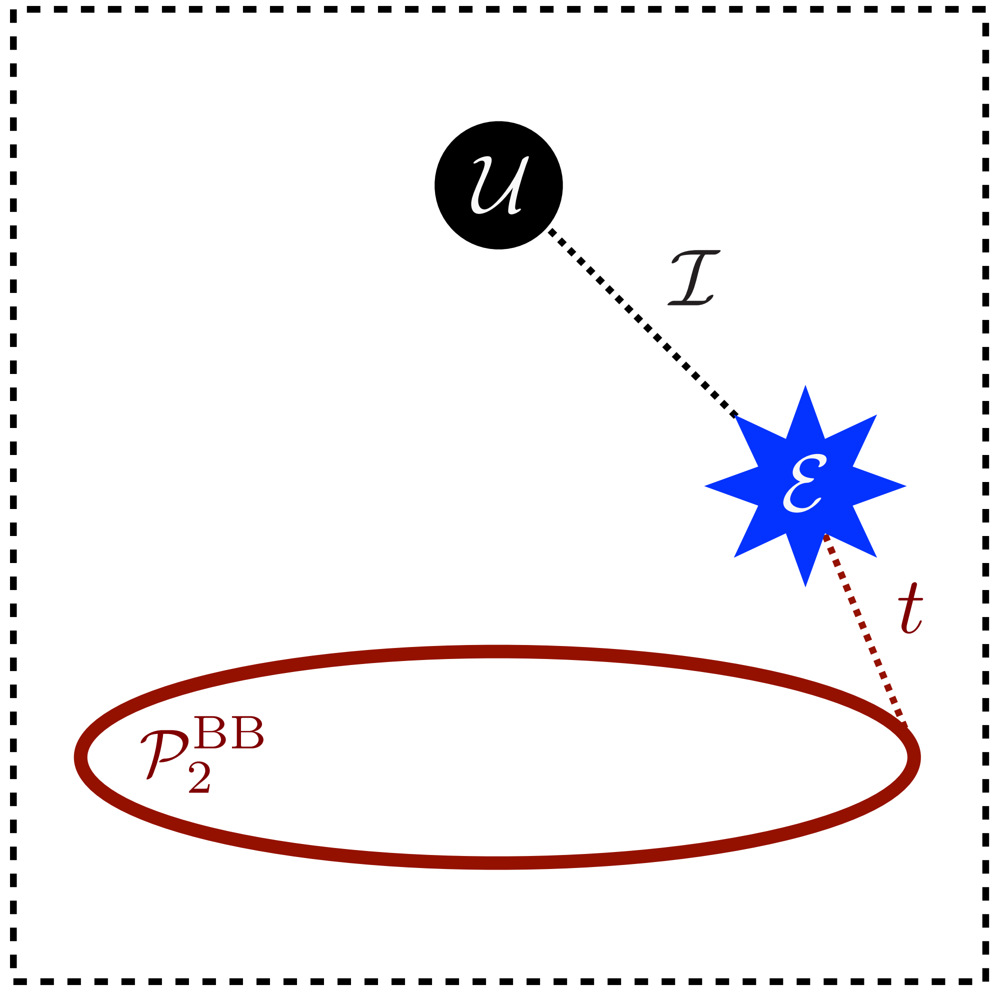
Defining Observables via Event Space Geometry

$$\boxed{\mathcal{O}(\mathcal{E}) = \min_{\mathcal{E}' \in \mathcal{M}} \text{EMD}_{\beta,R}(\mathcal{E}, \mathcal{E}')}$$

[PTK, Metodiev, Thaler, 2004.04159]

Thrust, spherocity, isotropy*

Distance of closest approach to a specific manifold



$$t(\mathcal{E}) = \min_{\mathcal{E}' \in \mathcal{P}_2^{\text{BB}}} \text{EMD}_2(\mathcal{E}, \mathcal{E}')$$

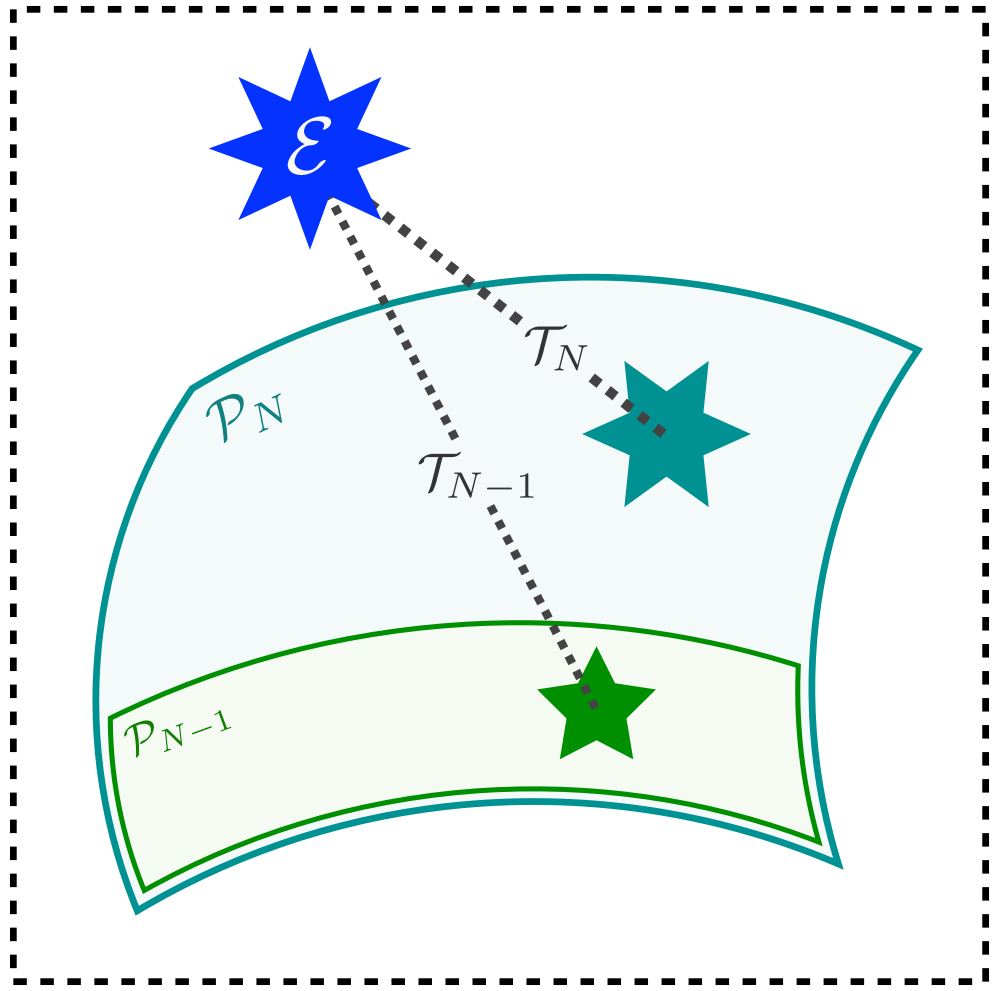
$$\sqrt{s(\mathcal{E})} = \min_{\mathcal{E}' \in \mathcal{P}_2^{\text{BB}}} \text{EMD}_1(\mathcal{E}, \mathcal{E}')$$

$$\mathcal{I}^{(\beta)}(\mathcal{E}) = \min_{\mathcal{E}' \in M_{\mathcal{U}}} \text{EMD}_\beta(\mathcal{E}, \mathcal{E}')$$

[Farhi, PRL 1977; Georgi, Machacek, PRL 1977]
*New! [Cesarotti, Thaler, 2004.06125]

N-jettiness

Minimum distance from event to N-particle manifold



without beam region

$$\mathcal{T}_N^{(\beta)}(\mathcal{E}) = \min_{\mathcal{E}' \in \mathcal{P}_N} \text{EMD}_\beta(\mathcal{E}, \mathcal{E}')$$

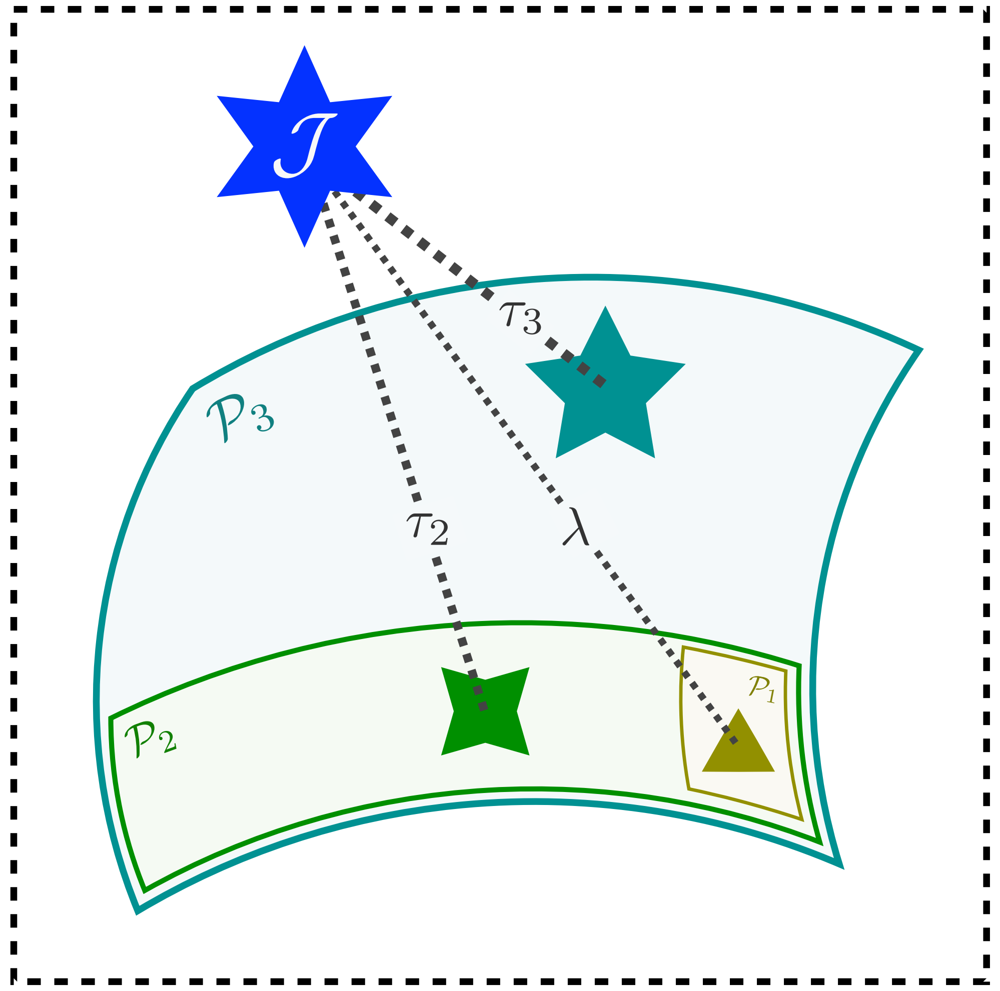
with constant beam distance R^β

$$\mathcal{T}_N^{(\beta,R)}(\mathcal{E}) = \min_{\mathcal{E}' \in \mathcal{P}_N} \text{EMD}_{\beta,R}(\mathcal{E}, \mathcal{E}')$$

[Brandt, Dahmen, Z. Phys 1979;
Stewart, Tackmann, Waalewijn, PRL 2010]

N-subjettiness, angularities

Smallest distance from jet to N-particle manifold



for recoil-free angularity

$$\lambda_\beta(\mathcal{J}) = \min_{\mathcal{J}' \in \mathcal{P}_1} \text{EMD}_\beta(\mathcal{J}, \mathcal{J}')$$

$$\tau_N^{(\beta)}(\mathcal{J}) = \min_{\mathcal{J}' \in \mathcal{P}_N} \text{EMD}_\beta(\mathcal{J}, \mathcal{J}')$$

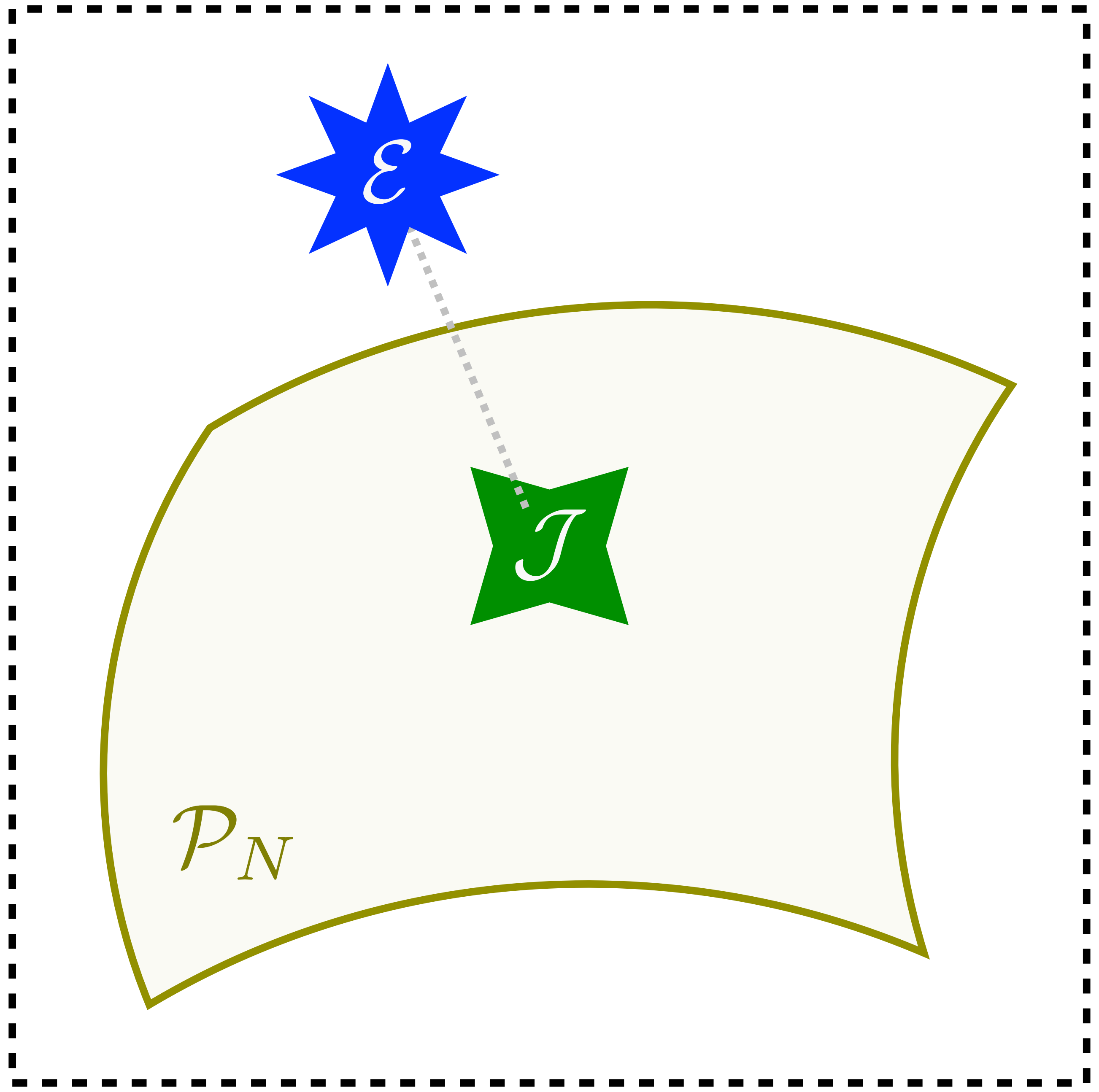
[Ellis, Vermilion, Walsh, Hornig, Lee, JHEP 2010;
Thaler, Van Tilburg, JHEP 2011, JHEP 2012]

Jets in the Space of Events – The Closest *N*-particle Description of an *M*-particle Event

[PTK, Metodiev, Thaler, 2004.04159]

Exclusive cone finding

XCone finds N jets by minimizing N-jettiness

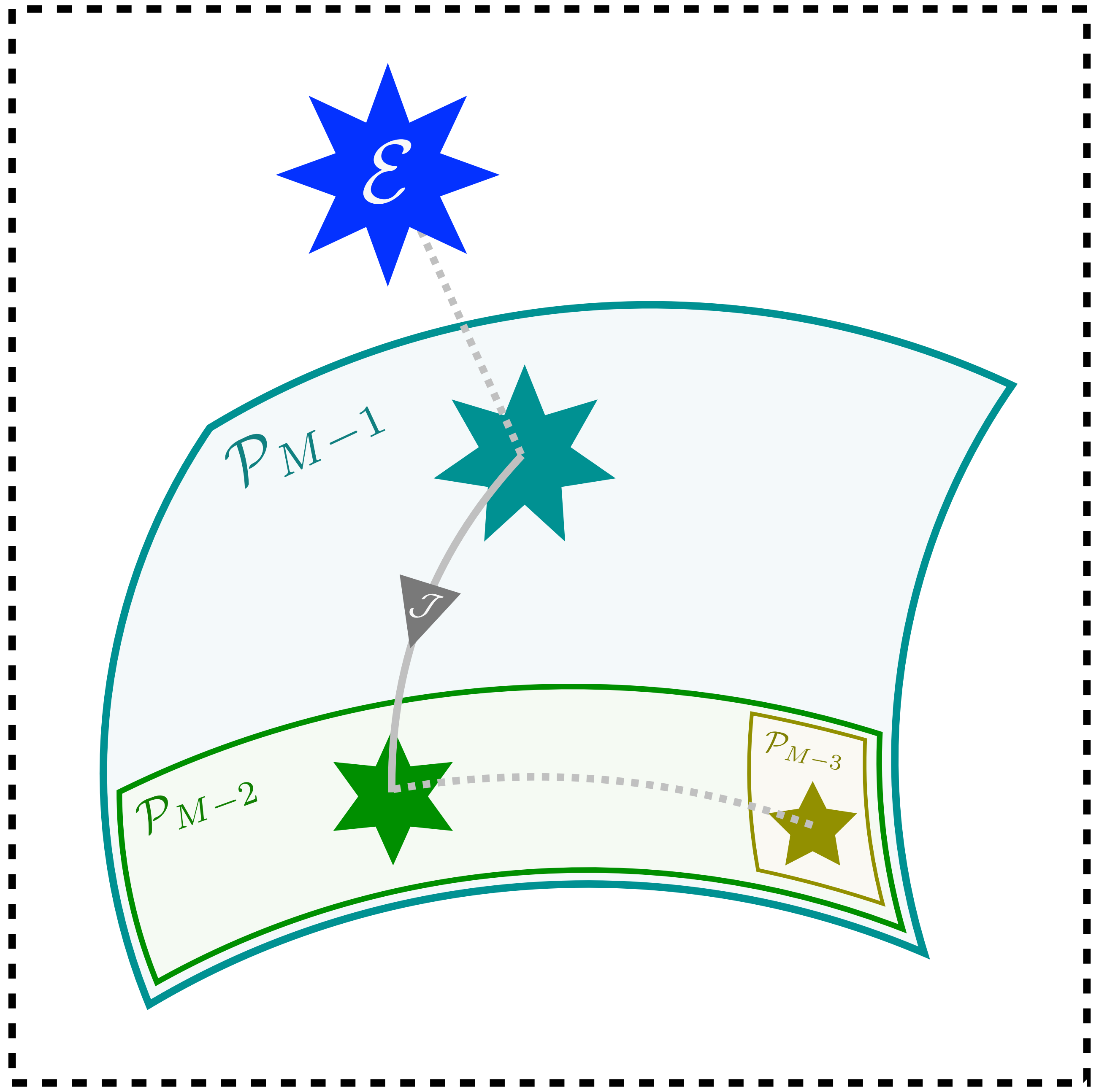


$$\mathcal{J}^{\mathrm{XCone}}_{N,\beta,R}(\mathcal{E}) = \underset{\mathcal{J}\in\mathcal{P}_N}{\arg\min}\, \mathrm{EMD}_{\beta,R}(\mathcal{E},\mathcal{J})$$

[Stewart, Tackmann, Thaler, Vermilion, Wilkason, JHEP 2015; Thaler, Wilkason, JHEP 2015]

Sequential recombination

Iteratively merges particles or identifies a jet



"destroying" energy corresponds to identifying a jet

event with one fewer particle after one step

$$\mathcal{E}^{(\beta,R)}_{M-1}(\mathcal{E}_M) = \underset{\mathcal{E}'_{M-1}\in\mathcal{P}_{M-1}}{\arg\min}\, \mathrm{EMD}_{\beta,R}(\mathcal{E}_M,\mathcal{E}'_{M-1})$$

[Catani, Dokshitzer, Seymour, Webber, Nucl. Phys. B 1993; Ellis, Soper, PRD 1993; Dokshitzer, Leder, Moretti, Webber, JHEP 1997; Cacciari, Salam, Soyez, JHEP 2008]

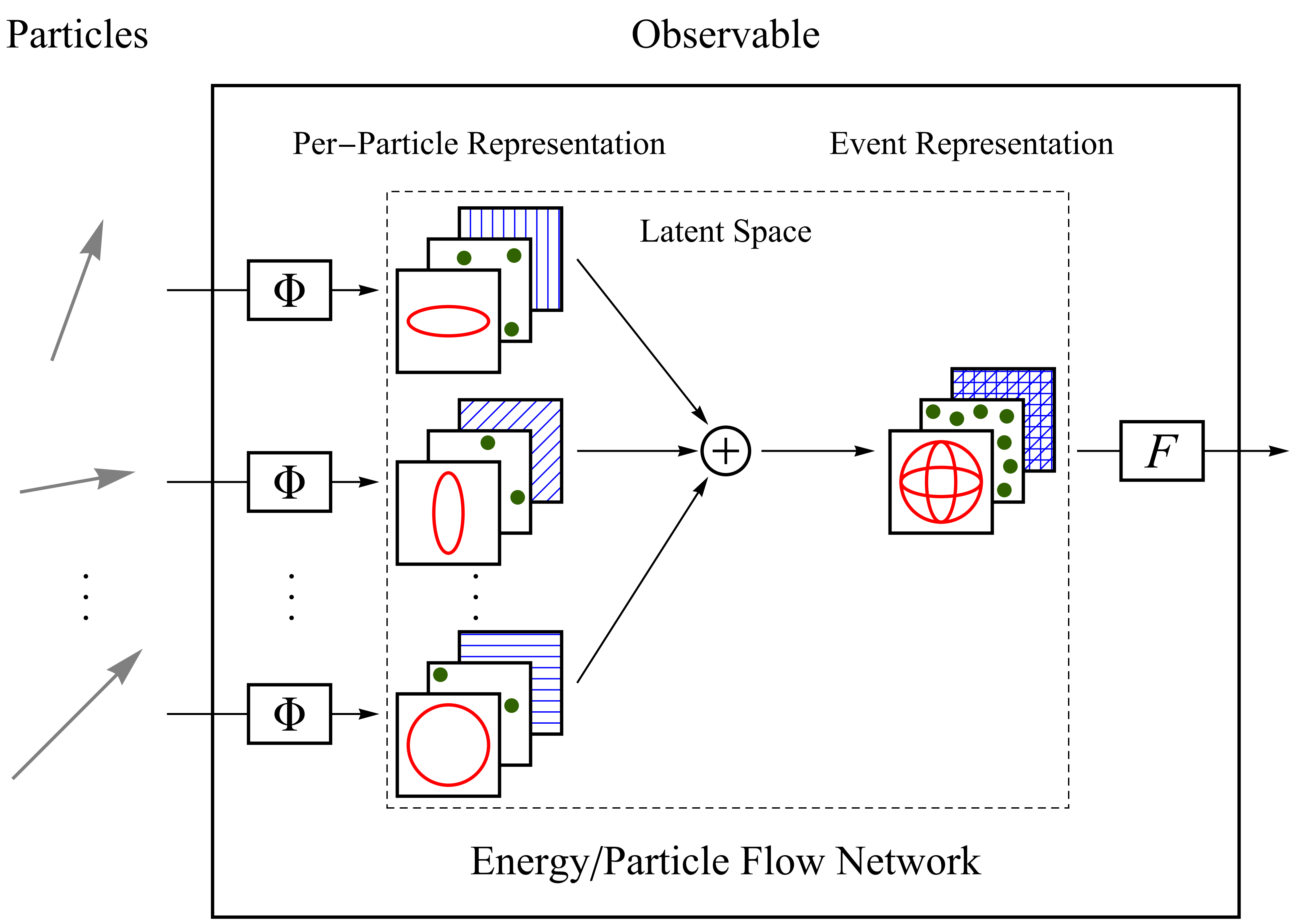
Energy Flow Networks (EFN)

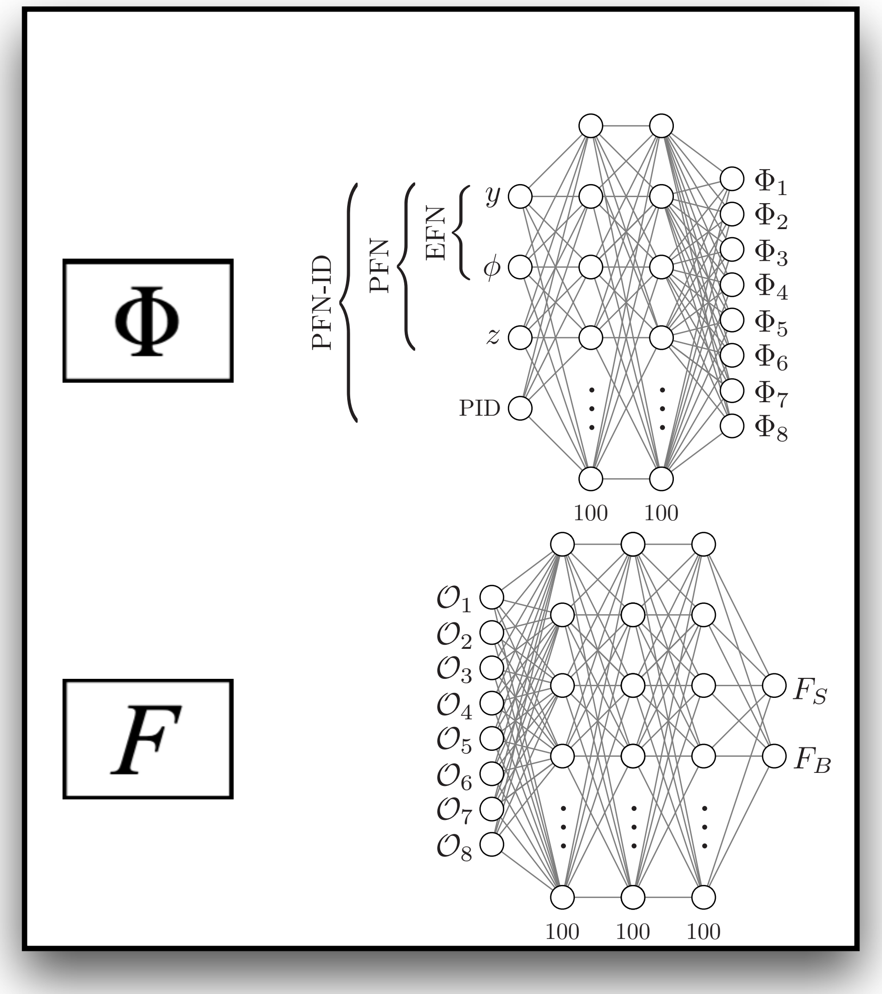
IRC safety is a statement of *linearity* in energy and *continuity* in geometry

Theorem: Any IRC-safe observable can be written in the following form:

$$f(\{p_1^\mu,\dots,p_M^\mu\}) = F\left(\sum_{i=1}^{M} z_i \vec{\Phi}(\hat{p}_i)\right), \quad \hat{p}_i = (y_i, \phi_i).$$

Proof: In 1810.05165. □





$$\text{EFN}: \quad \mathcal{O}_a = \sum_{i=1}^{M} z_i \Phi_a(y_i, \phi_i)$$

$$\text{PFN}: \quad \mathcal{O}_a = \sum_{i=1}^{M} \Phi_a(z_i, y_i, \phi_i, [\text{PID}_i])$$

Visualizing EFNs